THE HARD SQUARE ENTROPY CONSTANT

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Contents

1. Definition of the Constant

Let $g(m,n)$ be the number of $m \times n$ binary matrices which have no adjacent ones. That is, the number of matrices (m_{ij}) such that

$$m_{ij} = 1 \rightarrow m_{(i+1)(j)} = 0 \text{ and } m_{(i)(j+1)} = 0 \tag{1}$$

Define $c(m,n)$ to by

$$c(m,n) = g(m,n)^{\frac{1}{mn}} \tag{2}$$

As the size of the matrix approaches infinity, $c(m,n)$ approaches a limit.

$$\lim_{m,n\rightarrow\infty} c(m,n) = \kappa \tag{3}$$

This limit κ is known as the hard square entropy constant.

This constant is non-trivial to bound. Naive attempts to formulate bounds on it result in constant bounds that $1 \leq \kappa \leq 2$. These bounds do not improve as n increases.

2. History of the Constant

There is model of gasses used in statistical physics called the hard sphere gas model. One of the common simplifications to this model is to assume that all of the gas molecules lie at grid points in a plane and only interact with their four grid-neighbors. The grid is taken to be rigid and square, hence the "hard square" portion of the name.

R. J. Baxter, I. G. Enting, and S. K. Tsang wrote [2] in 1980. It was the first appearance of this constant. In that paper, they refered to κ as "the partition function per site." The thrust of their paper was to calculate a different physical constant, and thus they did not spend much time on the hard square entropy constant.

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$$
\begin{array}{l}
\vec{v}_1 = (0,0,0,0)\\
\vec{v}_2 = (0,0,0,1)\\
\vec{v}_3 = (0,0,1,0)\\
\vec{v}_4 = (0,1,0,0)\\
\vec{v}_5 = (0,1,0,1)\\
\vec{v}_6 = (1,0,0,0)\\
\vec{v}_7 = (1,0,0,1)\\
\vec{v}_8 = (1,0,1,0)
\end{array}
\qquad
T = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0\\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1\\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

FIGURE 1. Permissible length four vectors and the corresponding transfer matrix

According to [4], Prodinger and Tichy called $g(m,n)$ the Fibonacci number of a matrix. Weber showed that if $mn > 1$, then

$$1.45^{mn} < g(m,n) < 1.74^{mn} \tag{4}$$

In 1990, Konrad Engel wrote [4]. Engel formulated the problem in terms of sets of ordered pairs. Adapted to his notation, the function $g(m,n)$ is expressed as follows:

$$
\begin{aligned}
Z_{m,n} &= \{(i,j)\,|1 \le i \le m,\ 1 \le j \le n\}\\
A_{m,n} &= \{A \subseteq Z_{m,n}\,|(i_1,j_1),(i_2,j_2) \in A \to |i_1 - i_2| + |j_1 - j_2| \ne 1\}\\
g(m,n) &= |A_{m,n}|
\end{aligned} \tag{5}
$$

Engel went on to place bounds on the hard square entropy constant.

Engel employed what is known as the Corner Transfer Matrix method. In the Corner Transfer Matrix method, one considers binary vectors of length m which have no adjacent ones. There are F_{m+2} such vectors where F_{m+2} is the $(m+2)$-th Fibonacci number.

It is easy to see how the Fibonacci numbers came into the picture. Consider the permissible vectors of length m. If the vector starts with zero, then the other $m-1$ elements can be any permissible vector of length $m-1$. If the vector starts with one, the second element must be zero and the remaining $m-2$ elements can be any permissible vector of length $m-2$. Thus,

$$g(m,1) = g(m-1,1) + g(m-2,1) \tag{6}$$

subject to these starting conditions:

$$
\begin{aligned}
g(1,1) &= 2\\
g(2,1) &= 3
\end{aligned} \tag{7}
$$

It is clear that $g(m,1) = F_{m+2}$.

Next, one labels all of these vectors $\vec{v}_i$. One then constructs the transfer matrix of all of these binary vectors. The entry t_{ij} in the transfer matrix is one if the vector $\vec{v}_i$ is orthogonal to the vector $\vec{v}_j$ and zero otherwise. The permissible vectors of length four and the corresponding transfer matrix are shown in figure 1.

Engel used the largest eigenvalues of these transfer matrices to bound κ. Engel showed that if λ_m is the largest eigenvalue of the transfer matrix for binary vectors of length m, then

$$\frac{\lambda_{(2\ell)}}{\lambda_{(2\ell-1)}} \leq \kappa \leq \lambda_k^{\frac{1}{k}} \tag{8}$$

Engel used these results to obtain the following bounds on κ:

$$1.503 \leq \lim_{n\to\infty} g(n,n)^{1/n^2} \leq 1.514. \tag{9}$$

Engel further conjectured that

$$g(m,2k)^2 \geq g(m,2k-2)g(m,2k+2) \tag{10}$$

From this, one can prove that:

$$\frac{g(m,2)}{g(m,1)} < \frac{g(m,4)}{g(m,3)} < \frac{g(m,6)}{g(m,5)} < \ldots < \lambda_m < \ldots < \frac{g(m,5)}{g(m,4)} < \frac{g(m,3)}{g(m,2)} < \frac{g(m,1)}{g(m,0)} \tag{11}$$

If this were true, it would place a much tighter upper bound on κ.

In [3], Calkin and Wilf extended Engel's Corner Transfer Matrix methods to provide tighter bounds. They made the observation that the checkerboard pattern in an $n \times n$ matrix is a valid pattern with $2^{n^2/2}$ subsets. Using this, and several other geometric arguments, they strengthened the bounds given by Engel.

Baxter employed Calkin and Wilf's bounds to calculate the hard square entropy constant to forty three decimal places in [1]:

$$\kappa = 1.5030480824753322643220663294755536893857810 \tag{12}$$

3. Alternative Formulations

McKay, in [6], gave an alternative proof of bounds weaker than those of [3]. McKay used arguments based on legal configurations of integer multiples of known matrices. McKay defined $c(m,n)$ to be $g(m,n)^{\frac{1}{mn}}$. Then, he proved that for integers m, n, $a(m+1)$, and $b(n+1)$, where $a,b \geq 1$ (though a and b need not be integers), that

$$c(m,n)^\theta \leq c(a(m+1),b(n+1)) \leq 2^{1-\theta}c(m,n)^\theta \tag{13}$$

where

$$\theta = \frac{\lfloor a \rfloor}{a}\frac{\lfloor b \rfloor}{b}\frac{m}{m+1}\frac{n}{n+1}. \tag{14}$$

Although this formulation has weaker bounds than [3], it allows one to evaluate the limiting behaviour of a greater variety of permissible configurations.

The quantity $g(m,n)$ can be expressed in terms of graphs as the number of independent sets of vertices in an $(m+1)\times(n+1)$ grid graph.

It can also be expressed in game terms as the number of configurations of non-attacking princes on an $m \times n$ board. A prince is a piece which can move one space horizontally or vertically.

Additionally, it can be formulated as the number of walks of length n in orthogonal directions amongst the permissible vectors in $g(m,1)$. In fact, it is this formulation which permits the Corner Transfer Matrix method above. The transfer matrix is simply the adjacency matrix A of this graph. The number of possible walks is

$$\vec{\mathbf{1}}^T A^n \vec{\mathbf{1}} \tag{15}$$

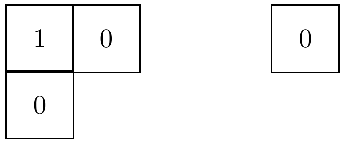

Figure 2. Pieces used to tile the quarter plane

And, it can be formulated as a tiling problem as the number of ways one can tile a quarter plane using the pieces in figure 2 without rotating the pieces.

The author attempted to formulate tighter bounds for κ based on the previous two methods. However, at his best, he achieved only [3]'s lower bound and the trivial upper bound 2.

References

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