SOME FOUNDATIONS OF ALGEBRAIC TOPOLOGY

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Abstract. This paper is a semi-formal summary of some basics of algebraic topology. It covers homotopy theory through the fundamental groups. It covers homology theory to the point of exact sequences.

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1. Motivation

I have had an interest in topology since second grade when I stumbled upon an entry for it in an encyclopedia. In recent years, I have become obsessed with understanding why there are 28 diffeomorphic structures on S^7. This paper represents the first small step toward that understanding. It is a summary of the beginnings of homotopy theory and homology theory. I will use many examples because examples really help me to put the concepts into practice.

I hope that most of the way that I present things will be accessible even to those who have not studied much topology. When I started reading [GH], it took me four or five weeks to follow everything presented in the first seventeen pages. For most of the other books that I looked at, I had to jump far into the book to get to information about homotopies and homologies. My primary goal in writing this paper is to write something that, if I had had it to start with, would have gotten me through the first seventeen pages of [GH] in under a week and through the rest of what I covered in [GH] in under three weeks. I think it's all that accessible. It's just not often presented verbosely.

With the exception of the (non-rigorous) notion of path, everything that I learned this quarter about algebraic topology was absolutely new to me. Through this quarter, I followed along [GH] through chapter 14. I had hopes of getting further, but I am quite pleased with all that I have learned. I owe much of my understanding to numerous discussions and emails with Dr. Meadows. Additionally, I utilized many other texts to help clarify the material in [GH].

2. Topologies

A topology is a set X together with a collection of subsets called open sets. The open sets can be defined in any manner at all so long as they obey these basic properties:

(1) $\emptyset$ and X are both open sets
(2) If U and V are open sets, then $U \cap V$ is an open set
(3) Any union (finite or not) of open sets is an open set

An easy example is the discrete topology. In the discrete topology, the open sets are the members of the power set. So, given a discrete set X, the open sets which define the discrete topology on X are the elements of $\mathcal{P}(X)$. So, for example, $\{x, z\}$ is one open set in the discrete topology on the set $X = \{x, y, z\}$.

Another example of a topology on a set is the indiscrete topology. The indiscrete topology on a set X has only $\emptyset$ and X as open sets.

Note. A topology on a set X is sometimes refered to as "the topological space X."

The usual topology on the real line $\mathbf{R}$ is composed of the open intervals on the real line and arbitrary unions of open intervals and finite intersections of open intervals. So, given four points along the real line, with $a \leq b \leq c \leq d$ then the set

$$\{r \in \mathbf{R} | a < r < b \text{ or } c < r < d\}$$

is one example of an open set in the usual topology on $\mathbf{R}$. And, of course $\emptyset$ and $\mathbf{R}$ are both open sets.

2.1. **Relative Topologies.** The usual topology on the unit interval I is the relative topology from $\mathbf{R}$. The relative topology on a subset is easily obtained from the topology on the set by restricting the open sets to the subset. If U is an open set in the topology on $\mathbf{R}$, then $I \cap U$ is an open set in the topology on I.

2.2. **Product Topologies.** If one has two topologies, one on X and one on Y, one can define the product topology on $X \times Y$. If x is in the set X and y is in the set Y, then (x, y) is in the set $X \times Y$. This association naturally extends to the topologies on the sets X and Y. If U is an open set in the topology on X and V is an open set in the topology on Y, then (U, V) is an open set in the topology on $X \times Y$.

So, if we take the discrete topology on $X = \{x, y, z\}$ as we did in section 2 and the usual topology on $\mathbf{R}$ as we did in section 2, then here are some examples of open sets in $X \times \mathbf{R}$:

$$(\{x, z\}, \{r \in \mathbf{R} | a < r < b \text{ or } c < r < d \text{ where } a \leq b \leq c \leq d\})$$

$$(\emptyset, \emptyset)$$

$$(X, \emptyset)$$

$$(\{y\}, \{r \in \mathbf{R} | r > 0\})$$

It's easy to see that product topologies associate, so that if we have topologies on X, Y, and Z, we have that the topologies $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are identical. It is abbreviated $X \times Y \times Z$.

3. Paths

Homotopy theory builds up from paths. Formally, a path in a topological space X is a continuous map $f : I \to X$ that takes the unit interval $I = [0, 1]$ into the topological space X.

Now, a function f from a topological space X into a topological space Y is said to be continuous if the inverse image of any open set in Y is an open set in X.

The discrete topology provides easy examples. Let f be a function from the discrete topology on $X = \{x, y, z\}$ to the discrete topology on $Y = \{1, 2\}$.

And, let:

$$f(a) = 1$$
$$f(b) = 1$$
$$f(c) = 2$$

We can easily check that:

$$f^{-1}(\emptyset) = \emptyset \in \mathcal{P}(X)$$
$$f^{-1}(\{1\}) = \{x, y\} \in \mathcal{P}(X)$$
$$f^{-1}(\{2\}) = \{z\} \in \mathcal{P}(X)$$
$$f^{-1}(\{1, 2\}) = \{x, y, z\} \in \mathcal{P}(X)$$

Another easy example of a path is the function $f : I \to I$ defined as $f(x) = x$. It is easy to see that the inverse image of any open set is an open set.

The function $f : I \to I$ defined as

$$f(x) = \begin{cases} 0 & ; x < \frac{1}{2} \\ 1 & ; x \geq \frac{1}{2} \end{cases}$$

is *not* continuous. We can see this because the inverse image of the open set $\frac{1}{2} < x \leq 1$ in I is the set $\frac{1}{2} \leq x \leq 1$. That set is not in the topology on I. It cannot be obtained from open intervals in $\mathbf{R}$ intersected with I through arbitrary unions and finite intersections.

Note. The "finite" is important here. The set can be obtained by the intersection of the infinite family of sets given by

$$S_n = \left\{ x \in I \,\middle|\, x > \left(\frac{1}{2} - \frac{1}{2^n} \right) \right\}$$

for integer $n \geq 2$.

3.1. **Loops.** Loops are a special class of paths that will be important to us in section 4.4 when we talk about fundamental groups. Quite unsurprisingly, a loop is simply a path whose endpoints are identical. Formally, a loop in X is a continuous function $f : I \to X$ with $f(0) = f(1)$.

A simple example of a loop is $f : I \to I$ given by

$$f(x) = \begin{cases} 2x & ; x < \frac{1}{2} \\ 2 - 2x & ; x \geq \frac{1}{2} \end{cases}$$

Another example of a loop is $f : I \to \mathbf{R}$ given by

$$f(x) = \sin(k\pi x)$$

where k is an integer.

Note. Even the trivial case where $k = 0$ is a loop. The constant path $f(x) = x_0$ is a useful loop. It will come up again in section 4.1.

3.2. **Multiplication of Paths.** If σ and τ are two paths in X such that the endpoint of σ is the starting point of τ (that is to say that $\sigma(1) = \tau(0)$), then we can define the multiplication of these paths as follows:

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & ; 0 \leq t < \frac{1}{2} \\ \tau(2t-1) & ; \frac{1}{2} \leq t \leq 1 \end{cases}$$

This is simply a function which traverses the path σ on the first half of I and then traverses the path τ on the second half of I. Because $\sigma(1) = \tau(0)$, $\sigma\tau\left(\frac{1}{2}\right)$ doesn't cause any problems with continuity. So, $\sigma\tau$ is a continuous function from I into X. In other words, $\sigma\tau$ is a path.

Note. Just because $\sigma(1) = \tau(0)$ does not necessarily mean that $\tau(1) = \sigma(0)$. So, even if we can multiply $\sigma\tau$, we may not be able to multiply $\tau\sigma$. An example of this is with σ and τ both paths in $\mathbf{R}$ where $\sigma(t) = 0$ and $\tau(t) = t$. Here, $\sigma\tau$ is a path. But, $\tau\sigma$ is not a path because the function

$$\tau\sigma(t) = \begin{cases} 2t & ; 0 \leq t < \frac{1}{2} \\ 0 & ; \frac{1}{2} \leq t \leq 1 \end{cases}$$

is not continuous. For example, the inverse image of the open set

$$\left\{ t \,\middle|\, t < \frac{1}{2} \right\}$$

is the set

$$\left\{ t \,\middle|\, t < \frac{1}{4} \text{ or } t \geq \frac{1}{2} \right\}$$

which is not an open set in I. In section 4.3, we will be multiplying loops so this worry goes away.

4. Homotopy Groups

4.1. **Homotopies.** To define homotopies, we will consider two paths σ and τ in a topological space X. We are only interested in paths which have the same endpoints. In other words, $\sigma(0) = \tau(0) = x_0$ and $\sigma(1) = \tau(1) = x_1$. In section 3, we saw that both σ and τ are continuous functions from the unit interval I into X. A homotopy is a continuous function F from the space $I \times I$ (we can think of the first I as for the paths, the second I as for the homotopy) into the topological space X that has the following characteristics:

$$F(s, 0) = \sigma(s)$$
$$F(s, 1) = \tau(s)$$

If we also have these conditions:

$$F(0, t) = \sigma(0) = \tau(0) = x_0$$
$$F(1, t) = \sigma(1) = \tau(1) = x_1$$

then we say that σ and τ are homotopic with endpoints held fixed.([GH] p. 6) Intuitively, a homotopy with endpoints held fixed defines a smooth deformation from one path to another but doesn't move the endpoints of the paths in the process.

We can think of this functions F in a few different ways that help us visualize what it actually means. For example, we could consider the family of functions $F_{(*,t)} : s \to F(s,t)$. By this notation, I mean that, at any given t, there is a function $F_{(*,t)}$ which is a function of s and is equivalent at every s to $F(s,t)$. Each of these $F_{(*,t)}$ is a path from x_0 to x_1 in X. Here, there are the note-worthy paths $F_{(*,0)} = \sigma$ and $F_{(*,1)} = \tau$.

We could instead consider the family of functions $F_{(s,*)} : t \to F(s,t)$. Each of these functions is a path from $\sigma(s)$ to $\tau(s)$ in X. Here, there are the note-worthy constant loops $F_{(0,*)} = x_0$ and $F_{(1,*)} = x_1$. Because $F_{(0,*)}$ and $F_{(1,*)}$ are constant loops, we say that this homotopy relates σ and τ relative to $\{0,1\}$ or simply that σ and τ are homotopic relative to $\{0,1\}$. Notationally, this is:

$$\sigma \simeq \tau \text{ rel } \{0,1\}$$

Pictorially, a homotopy with endpoints fixed looks something like this:

$$\begin{tikzcd}
x_1 \arrow[r, equal] & x_1 \\
x_0 \arrow[u, "\sigma"] \arrow[r, equal] & x_0 \arrow[u, "\tau"']
\end{tikzcd}$$

Where each $F_{(s,*)}$ runs horizontally across the picture from left to right and each $F_{(*,t)}$ runs vertically across the picture from bottom to top.

There may be other s for which $F_{(s,*)}$ are constant loops. If that is the case, then we say that σ and τ are homotopic relative to whatever set of s's have $F_{(s,*)}$ as constant loops. For example, if $F(s,t) = \sigma(s) = \tau(s)$ for all $s \in S$, then $\sigma \simeq \tau$ rel S.

Here is an example of two homotopic paths in $\mathbf{R}$. Let $\sigma(s) = s$ and $\tau(s) = s^2$. We can define the function:

$$F(s,t) = (1-t) \cdot s + t \cdot s^2$$

From this, we can easily see that this function defines a homotopy with endpoints fixed between σ and τ because:

$$F(s,0) = \sigma(s)$$

$$F(s,1) = \tau(s)$$

$$F(0,t) = \sigma(0) = \tau(0) = 0$$

$$F(1,t) = \sigma(1) = \tau(1) = 1$$

Here is another example of a homotopy. One can easily see that if σ is a path from x_0 to x_1, then the path τ given by $\bar{\sigma}(s) = \sigma(1-s)$ simply follows the path σ in the opposite direction from x_1 to x_0. We can show that the loop $\sigma\bar{\sigma}$ (that is, the loop that follows the path σ followed by the path $\bar{\sigma}$) is homotopic to the constant loop $C_{x_0}(s) = x_0$. To do this, we define:

$$F(s,t) = \begin{cases} \sigma\big((1-t)\cdot(2s)\big) & ; 0 \leq s < \frac{1}{2} \\ \bar{\sigma}\big((1-t)\cdot(2s-1)\big) & ; \frac{1}{2} \leq s \leq 1 \end{cases}$$

This defines a homotopy with endpoints fixed from the product $\sigma\bar{\sigma}$ to the constant loop C_{x_0}.

4.2. **Homotopic Maps.** We can extend this idea of homotopy beyond just paths. A path is a continuous function from I into X. If, instead, we had continuous functions f and g from Y into X, then if we can define a homotopy $F : Y \times I \to X$ with the properties that $F(y,0) = f(y)$ and $F(y,1) = g(y)$, it is a homotopy between the maps f and g. And, if A is the set of all y such that $F(y,t)$ remains constant over all $t \in I$, then

$$f \simeq g \text{ rel } A$$

4.3. **Homotopy Equivalence Classes.** We can easily verify that the relationship "homotopic with endpoints held fixed" is an equivalence relationship. The homotopy $F : I \times I \to X$ given by $F(s,t) = \sigma(s)$ shows that $\sigma \simeq \sigma$ rel $\{0,1\}$ (in fact, it is homotopic to itself relative to all of I). This shows that the relationship is reflexive. If $F(s,t)$ is a homotopy from σ to τ relative to $\{0,1\}$, then the function $\bar{F}(s,t) = F(s,1-t)$ is a homotopy from τ to σ relative to $\{0,1\}$. This shows that the relationship is symmetric. And if $F_{\sigma,\tau}$ is a homotopy from σ to τ relative to $\{0,1\}$ and $F_{\tau,\rho}$ is a homotopy from τ to ρ relative to $\{0,1\}$, then we can define

$$F_{\sigma,\rho} = \begin{cases} F_{\sigma,\tau}(s,2t) & 0 \leq t < \frac{1}{2} \\ F_{\tau,\rho}(s,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

which is a homotopy from σ to ρ relative to $\{0,1\}$. This shows that the relationship is transitive.

Because of this, we can use homotopies with endpoints held fixed to form equivalence classes of paths. The equivalence class of a path σ is denoted $[\sigma]$.

We can multiply these equivalence classes in the same way that we multiplied paths in section 3.2 by definining $[\sigma][\tau]$ to be $[\sigma\tau]$. We can show that this is true because if F is a homotopy from $\sigma \simeq \sigma'$ rel $\{0,1\}$ and G is a homotopy from $\tau \simeq \tau'$ rel $\{0,1\}$, then every $F_{(*,t)}$ is a path from $\sigma(0)$ to $\sigma(1)$ and every $G_{(*,t)}$ is a path from $\tau(0)$ to $\tau(1)$. We can multiply *those* paths $F_{(*,t)}G_{(*,t)}$ for every $t \in I$. In that process, then, we have defined a homotopy from $\sigma\tau \simeq \sigma'\tau'$ rel $\{0,1\}$ which shows that $[\sigma\tau] = [\sigma'\tau']$.

Pictorially, F looks like this:

$$F = \begin{tikzcd} \sigma(1) \arrow[r, equal] & \sigma'(1) \\ \sigma(0) \arrow[u, "\sigma"] \arrow[r, equal] & \sigma'(0) \arrow[u, "\sigma'"'] \end{tikzcd} \quad G = \begin{tikzcd} \tau(1) \arrow[r, equal] & \tau'(1) \\ \tau(0) \arrow[u, "\tau"] \arrow[r, equal] & \tau'(0) \arrow[u, "\tau'"'] \end{tikzcd}$$

with each $F_{(*,t)}$ and each $G_{(*,t)}$ running vertically from bottom to top.

Pictorially, FG then looks like this:

$$\begin{tikzcd} \tau(1) \arrow[r, equal] & \tau'(1) \\ \sigma(1) = \tau(0) \arrow[u, "\tau"] \arrow[r, equal] & \sigma'(1) = \tau'(0) \arrow[u, "\tau'"'] \\ \sigma(0) \arrow[u, "\sigma"] \arrow[r, equal] & \sigma'(0) \arrow[u, "\sigma'"'] \end{tikzcd}$$

with each $F_{(*,t)}G_{(*,t)}$ running vertically from bottom to top.

4.4. **Fundamental Groups.** For the fundamental group of a topological space X, we are concerned with the equivalence classes of loops which are rooted at a given point x_0 in our topological space X. The equivalence classes are exactly the ones described in section 4.3 for the paths σ in X where $\sigma(0) = \sigma(1) = x_0$.

As we hinted in section 3.1, the fact that we are using loops that are rooted at a common point alleviates any concern that we will have trouble multiplying the equivalence classes of these paths.

The fundamental group of a space X at the point x_0 is the set of the equivalence classes of loops rooted at x_0. The multiplication in that group is the multiplication of the equivalence classes defined in section 4.3. The fundamental group is denoted $\pi_1(X, x_0)$. The example in section 4.1 which showed that if σ is a path that starts at x_0, then $\sigma\bar{\sigma}$ where $\bar{\sigma}$ is defined by $\bar{\sigma}(s) = \sigma(1-s)$ is in the same equivalence class as the constant loop at x_0 (which we'll abbreviate C_{x_0}. In the fundamental group, the equivalence class of the constant loop is the identity element in the group. It is easy to verify that any loop at x_0 multiplied by the constant loop at x_0 is homotopic to the original loop. So, the inverse of the equivalence class of a loop σ is simply the equivalence class of the reverse loop $\bar{\sigma}(s) = \sigma(1-s)$ because $[\sigma\bar{\sigma}] = [C_{x_0}] = [\bar{\sigma}\sigma]$.

If the topological space X is path-connected (meaning that one can construct a path between any two points in X), then the choice of x_0 is irrelevant. If σ and σ' are loops at x and τ is a path from x_0 to x, then the paths $\tau\sigma\bar{\tau}$ and $\tau\sigma'\bar{\tau}$ are loops at x_0 which are homotopic if and only if σ and σ' are homotopic. Because of this, $\pi_1(X, x_0)$ is sometimes abbreviated to $\pi_1(X)$ when X is path-connected.

4.4.1. *The Circle.* Now, lets spend a bit of time looking at the fundamental group of the circle S^1. So, pick a point on the circle and call it 1. We can use our intuitive notion of a homotopy as a smooth deformation of one path to another to get a feel for the fundamental group of the circle. If we consider the equivalence class of loops at 1 that are homotopic to the constant loop $C_1(s) = 1$, we can see that a path which goes part way around the circle before coming back to 1 is homotopic to C_1. And, we can see that a path which goes around the circle one and a quarter times and then goes back around the circle one and a quarter times in the other direction and stops at 1 is homotopic to C_1. With our intuitive notion of homotopies, we can be reasonably certain that any path that passes through 1 and even number of times (but may be tangent to it any number of times) with half of the passes in one direction and half in the other direction is homotopic to C_1. Another way to conceptualize this is that any continuous function from $f : I \to \mathbf{R}$ with $f(0) = f(1) = 0$ defines a path in S^1 which is homotopic to C_1 where a point $f(x) = y$ is y units around the circle in the counter-clockwise direction.

Note. The choice of counter-clockwise instead of clockwise is entirely arbitrary, but it will be simpler to use counter-clockwise in the following development.

Above, we "unwound" every time we "wound" around the circle. Intuitively, if we simply wind around the circle once and stop at 1 when we come upon it again, there is no way we can keep the endpoints of our loop fixed and deform our loop into C_1. So, we would guess that this loop is not in the same equivalence class as C_1.

Now, let's make this intuitive conception more rigorous. If we take S^1 to be the set of all complex numbers with absolute value 1, then we have a homomorphism $\phi : \mathbf{R} \to S^1$ (where we're using only the additive structure of $\mathbf{R}$) defined by $\phi(x) = e^{2\pi x}$.([GH] p. 16)

There are two lemmas which are critical for what follows.

Lemma 4.1. ***The Lifiting Lemma.*** *If σ is a path in S^1 with initial point 1, then there exists a unique path $\hat{\sigma}$ in $\mathbf{R}$ with initial point 0 such that $\phi \circ \hat{\sigma} = \sigma$. (*[GH] *p. 16)*

Lemma 4.2. ***The Covering Homotopy Lemma.*** *If τ is a path in S^1 with initial point 1 such that F is a homotopy that provides this equivalence:*

$$\sigma \simeq \tau \ rel \ \{0, 1\}$$

then there is a unique $\hat{F} : I \times I \to \mathbf{R}$ such that $\hat{F}$ is a homotopy that provides this equivalence:

$$\hat{\sigma} \simeq \hat{\tau} \ rel \ \{0, 1\}$$

and $\phi \circ \hat{F} = F$. ([GH] *p. 16)*

We will prove more general versions of these lemmas in section 4.4.2. For the time being, we will only focus on what they mean.

Lemma 4.1 is saying that if we have a path $\sigma : I \overset{cont.}{\longrightarrow} S^1$ with $\sigma(0) = 1$, then there is a unique path $\hat{\sigma} : I \overset{cont.}{\longrightarrow} \mathbf{R}$ with $\hat{\sigma}(0) = 0$ that the function $\phi : \mathbf{R} \to S^1$ maps to σ. Diagramaticklly, this is:

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, "\hat{\sigma}"'] & I \arrow[d] \\ (\mathbf{R}, 0) \arrow[d, "\phi"'] & {} \arrow[d, "\sigma"] \\ (S^1, 1) \arrow[r, equal] & (S^1, 1) \end{tikzcd}$$

where $\hat{\sigma}$ is uniquely determined by σ. The path $\hat{\sigma}$ is called the lifting of the path σ from $(S^1, 1)$ to $(\mathbf{R}, 0)$. Saying it another way, the lifting of a path σ in S^1 that starts at the point 1 into the unique path in $\mathbf{R}$ starting at the point 0 is denoted $\hat{\sigma}$.

Note. All of this is given a fixed ϕ. The lifting is unique for a given ϕ.

Another way to say all of this is to say that ϕ defines a one-to-one correspondence between paths in S^1 which start at 1 and paths in $\mathbf{R}$ which start at 0.

Lemma 4.2 is saying that if we have a homotopy with endpoints held fixed between two paths σ and τ in S^1, then there is a unique homotopy with endpoints held fixed between the unique liftings $\hat{\sigma}$ and $\hat{\tau}$ of the paths σ and τ into $\mathbf{R}$. Another way to say all of this is to say that ϕ defines a one-to-one correspondence between homotopies in S^1 between paths which start at 1 and homotopies in $\mathbf{R}$ between paths which start at 0.

By lemma 4.1, under our mapping ϕ, any path σ in S^1 that starts at 1 and ends at 1 has a corresponding unique path in $\mathbf{R}$ that starts at 0 and ends at some point x where $\phi(x) = 1$. The only places where $\phi(x) = 1$ are where x is an integer.

Similarly, by lemma 4.2, if two paths σ and τ in S^1 that start at 1 and end at 1 are homotopic with endpoints held fixed, then their liftings $\hat{\sigma}$ and $\hat{\tau}$ into $\mathbf{R}$ starting at 0 are homotopic with endpoints held fixed. The particular bit of interest here is that if σ and τ are in the same equivalence class of loops in S^1, then the unique functions $\hat{\sigma}$ and $\hat{\tau}$ have the same endpoints. Particularly, $\hat{\sigma}(1) = \hat{\tau}(1) = x$ for some integer x.

Given all of this, we can define a map $\chi : \pi_1(S^1, 1) \to \mathbf{Z}$ using the endpoint of the lifting of σ as follows:

$$\chi\big([\sigma]\big) = \hat{\sigma}(1) \tag{1}$$

This mapping creates an isomorphism.([GH] p. 17)

First, we will show that χ is a homomorphism. If we take two equivalence classes $[\sigma], [\tau] \in \pi_1(S^1, 1)$, then we can let $m = \chi\big([\sigma]\big)$ and $n = \chi\big([\tau]\big)$. Both m and n will be integers. If we define $\hat{\rho}$ to be a path in $\mathbf{R}$ from m to $m + n$ by defining $\hat{\rho}(s) = m + \hat{\tau}(s)$, then we can easily see that $\hat{\sigma}\hat{\rho}$ is the lifting of

$\sigma\tau$ because the lifting of σ gets us to m and the lifting of τ takes over from there. We can also see that $\phi \circ \hat{\rho} = \tau$. So, $\chi\big([\sigma]\big) + \chi\big([\tau]\big) = \chi\big([\sigma][\tau]\big)$

Note. At first glance, it may look like $\hat{\tau}$ and $\hat{\rho}$ are two different liftings of τ into $\mathbf{R}$ violating lemma 4.1. They are distinct if $m \neq 0$, and they are both liftings of τ into $\mathbf{R}$. But, $\hat{\tau}$ begins at 0 and $\hat{\rho}$ begins at m. Lemma 4.1 only claims the lifting to a path which starts at 0 is unique.

Next, we will show that it is onto. Stated another way: given any integer n, we can find a loop σ whose equivalence class maps to n under χ. We can take the path $\hat{\sigma}$ in $\mathbf{R}$ given by $\hat{\sigma}(s) = s \cdot n$. Once we apply ϕ to this, we get our $\sigma = \phi \circ \hat{\sigma}$. By equation 1, we know that $\chi\big([\sigma]\big) = n$.

And, finally, we will show that it is one-to-one. Since χ is a homomorphism, it suffices to show that there is only one element in the kernel of χ. That is to say: there is only one equivalence class of loops in S^1 rooted at 1 that maps to 0 under χ. If some loop σ maps to 0 under χ, then the lifted path $\hat{\sigma}$ must have both endpoints at 0 in $\mathbf{R}$. In other words, $\hat{\sigma}$ must be a loop in $\mathbf{R}$ at 0. The function $\hat{F}(s,t) = (1-t) \cdot \hat{\sigma}(s)$ defines a homotopy with endpoints held fixed from $\hat{\sigma}$ to the constant loop at 0 in $\mathbf{R}$ (which we've been calling C_0). By lemma 4.2, we can apply ϕ to $\hat{F}$ to obtain a homotopy F with endpoints held fixed in S^1 for loops starting at 1. Now, because $\phi \circ C_0 = C_1$, we see that every loop rooted at 0 in $\mathbf{R}$ is homotopic to the constant loop C_0. This translates directly into any such loop in $\mathbf{R}$ has an image under ϕ that is in the same equivalence class as C_1. In other words, $[C_1]$ is the only element in the kernel of χ so χ is one-to-one.

Finally, we have proven that the fundamental group of the circle is isomorphic to the integers under addition.

4.4.2. *Covering Spaces.* In the previous section, we used the function $\phi : \mathbf{R} \to S^1$ in our development. There $\mathbf{R}$ and ϕ formed a covering space of S^1. A covering space of a topological space X is a topological space E together with a mapping $p : E \to X$ where the inverse image of any open set in X is a disjoint union of open sets S_i in E. The S_i are called sheets. And, for a given open set U, each S_i in $p^{-1}(U)$ is mapped homeomorphically onto the open set U.

That's a bunch to cover all at once. First, let's just say that in the previous section, the topological space $\mathbf{R}$ plays the part of the space E in this definition and that the function ϕ plays the part of the function p.

Next, we'll tackle homeomorphic. Two sets A and B are homeomorphic if there exists a function $f : A \to B$ that has a two-sided inverse f^{-1}. That is to say that $f^{-1}\big(f(a)\big) = a$ for all $a \in A$ and $f\big(f^{-1}(b)\big) = b$ for all $b \in B$. We say that f is a homeomorphism from A to B. In our case, p is a homeomorphism from each S_i in $p^{-1}(U)$ to U for any open set U in X.

Because the sheets S_i are disjoint, the inverse image of any $x \in X$ must be a discrete set. If the set $p^{-1}(x)$ has the same cardinality for all $x \in X$, then the we say that the covering space evenly covers X.

To understand this all better, we will use our example of $\phi : \mathbf{R} \to S^1$ given by $\phi(x) = e^{2\pi x}$. Let us consider the inverse image of this open set in S^1:

$$U = \left\{ a + bi \in S^1 \, | a > 0 \right\}$$

$\phi^{-1}(U)$ would be the union of the sets:

$$S_n = \left\{ x \,\middle|\, n - \frac{1}{4} < x < n + \frac{1}{4} \right\}$$

where n ranges over all of the integers. We can verify that the function

$$\phi_n^{-1}(a + bi) = \frac{\arcsin b}{2\pi} + n$$

(with $-\frac{\pi}{2} \leq \arcsin b < \frac{\pi}{2}$) forms a two-sided inverse for ϕ in mapping S_n onto U. Thus, each S_n is mapped homeomorphically onto U by ϕ. This same sort of construction can be made (albeit the arcsin is a bit flimsy for the task) for any open set in S^1. So, $\mathbf{R}$ and ϕ form a covering space of S^1. In fact, $\mathbf{R}$ and ϕ evenly covers S^1.

We need one more bit of notation before we can prove the lemmas from the previous section. The notation $(E, e_0) \xrightarrow{p} (X, x_0)$ is called a covering space with base points if E and p form a covering space of X and $p(e_0) = x_0$. Technically, in the previous section, we used a covering space with base points. We used $(\mathbf{R}, 0) \xrightarrow{\phi} (S^1, 1)$.

Theorem 4.1. *Unique Lifting Theorem. If we have a covering space with base points $(E, e_0) \xrightarrow{p} (X, x_0)$ which evenly covers X and some continuous map $f : (Y, y_0) \to (X, x_0)$ (with Y connected), then if a continuous map $\hat{f} : (Y, y_0) \to (E, e_0)$ exists such that $p \circ \hat{f} = f$, it is unique.*

To prove this, we will assume there is some map $\hat{f}_1 : (Y, y_0) \to (E, e_0)$ with $p \circ \hat{f}_1 = f$ and show that it must be identical to $\hat{f}$. We will define two disjoint sets:

$$A = \left\{ y \in Y \,\middle|\, \hat{f}(y) = \hat{f}_1(y) \right\}$$

$$D = \left\{ y \in Y \,\middle|\, \hat{f}(y) \neq \hat{f}_1(y) \right\}$$

So, we'll take a $y_1 \in Y$ and an open set U in the topological space X containing $f(y)$.

Now, if $y_1 \in A$, then $\hat{f}(y_1) = \hat{f}_1(y_1)$. This point $\hat{f}(y_1)$ lies on some sheet $S \in E$ which contains $p^{-1}(U)$. And, because $p^{-1}(U)$ is mapped homeomorphically onto U by p, we know that $p^{-1}(U)$ is an open set. And, because $\hat{f}$ and $\hat{f}_1$ are continuous, we know that $\hat{f}^{-1}(S)$ and $\hat{f}_1^{-1}(S)$ are open sets containing y_1. So, $\hat{f}^{-1}(S) \cap \hat{f}_1^{-1}(S)$ is an open set containing y_1 and contained in A.

However, if $y_1 \in D$, then $\hat{f}(y_1) \neq \hat{f}_1(y_1)$. So, $\hat{f}(y_1)$ lies in some sheet S_0 and $\hat{f}_1(y_1)$ lies in some other sheet S_1. And, again, $\hat{f}^{-1}(S_0)$ and $\hat{f}^{-1}(S_1)$ are open sets in Y containing y_1. So, $\hat{f}^{-1}(S_0)\cap\hat{f}^{-1}(S_1)$ is an open set containing y_1 and contained in D.

Because Y is connected, the only way that A and D could be disjoint, open sets that completely compose Y is that one must equal Y and the other must be empty. But, A contains at least the base point y_0. So, A cannot be empty. So, the set A must be the whole set Y. In other words $\hat{f}(y) = \hat{f}_1(y)$ for all $y \in Y$. So, $\hat{f} = \hat{f}_1$ which means that $\hat{f}$ is unique.

Theorem 4.2. *Path Lifting Theorem. For a covering space with base points $(E, e_0) \xrightarrow{p} (X, x_0)$ that evenly covers X, if σ is a path in X with initial point x_0, there is a unique path $\hat{\sigma}$ in E with initial point e_0 such that $p \circ \hat{\sigma} = \sigma$.*

By theorem 4.1, we know that such a path must be unique. So, if e_0 is on some sheet S and we let ψ be the inverse of p over S, then $\hat{\sigma} = \psi \circ \sigma$ is the lifted path. It is easy to see that $p \circ \psi \circ \sigma = \sigma$. This proves lemma 4.1.

Theorem 4.3. *Covering Homotopy Theorem. If we have a covering space with base points $(E, e_0) \xrightarrow{p} (X, x_0)$ which evenly covers X and an arbitrary continuous function $f : (Y, y_0) \to (X, x_0)$ which has a lifting $\hat{f} : (Y, y_0) \to (E, e_0)$, than any homotopy $F : Y \times I \to X$ with $F(y, 0) = f(y)$ for all $y \in Y$ can be lifted to a unique homotopy $\hat{F} : Y \times I \to E$ with $\hat{F}(y, 0) = \hat{f}(y)$ for all $yinY$.*

The important part of this theorem, for our purposes is that it implies that if σ and τ are paths in X that are homotopic with endpoints held fixed, then $\hat{\sigma}$ and $\hat{\tau}$ are paths in E which are homotopic with endpoints held fixed. This theorem, then subsumes the lemma 4.2.

4.4.3. *Other Common Spaces.* Without being rigorous, one can play around with paths in a surface and get a pretty good inkling about the fundamental group of the surface. For example, in $\mathbf{R}^n$ for all positive n, any loop can be continuously contracted to a point. So, the fundamental group of $\mathbf{R}^n$ is trivial (and written $\{0\}$). Similarly, the fundamental group of the spheres S^n with integer $n > 1$ are all trivial. The situation for the annulus, turns out to be much the same as the situation for S^1. The number of times around the hole in the annulus corresponds to the number of times around the circle. As a result, the fundamental group of the annulus is isomorphic to the integers under addition, just as S^1 was.

A more interesting example is the projective sphere P^2. Its fundamental group is isomorphic to the additive group $\mathbf{Z}_2$ of the integers modulo 2. One can make a loop that is not homotopic to the constant loop. But, any loop not homotopic to the constant loop is homotopic to this loop. And, as soon as one adds two such loops together, one gets a loop homotopic to the constant loop.

Another nice thing to note is that the fundamental group on a product space is the direct product of the fundamental groups of the spaces. So, for example, the fundamental group of the infinite cylinder $S^1 \times \mathbf{R}$ is $\mathbf{Z} \times \{0\}$. This also means that the fundamental group of the torus $S^1 \times S^1$ is $\mathbf{Z} \times \mathbf{Z}$.

4.5. **Loop Spaces.** The loop space of X at a point x_0 is denoted Ω_{x_0}. It is composed of all of the loops in X at x_0. So, for example, the loops $\sigma(s) = 0$ and $\tau(s) = 4s(1-s)$ in I are members of Ω_0 for I.

It is easy to show that if σ and τ are in the same path-connected component of Ω_{x_0}, then σ and τ are homotopic paths in X. If f is a path from σ to τ in Ω_{x_0}, then the function $F(s,t) = \big(f(s)\big)(t)$ is a homotopy in X between σ and τ. In other words, each path-connected component of Ω_{x_0} represents a member of $\pi_1(X, x_0)$.

I spent a long time trying to visualize loop spaces of simple spaces. Here is what I came up with. If x_0 is isolated (meaning there exists an open set which only contains x_0), then Ω_{x_0} has only one member—the constant loop $f(s) = x_0$.

But, if X is not discrete things get complicated quickly. For example, let's look at the loop space at 0 of the unit interval I. Every loop in I is homotopic to the constant loop $f(s) = 0$. So, how can we describe all of the loops in I? The loop space in I is set of all continous functions $f : I \times I \to I$ with $f(s,0) = f(s,1) = 0$. Very unrigorously, the x in $\sigma(s_0) = x$ can change slightly to get a new path τ. If the change is arbitrarily small, then σ and τ will be arbitrarily close in loop space. This sort of change can happen at every point s in I. So, the loop space of I is roughly I^2-ish.

4.6. **Higher Order Homotopy Groups.** The concept of loop spaces from the previous section is used to create higher order homotopy groups. The fundamental group of a space X rooted at the point x_0 is denoted $\pi_1(X, x_0)$. I was immediately curious why the subscript 1 was needed. The higher order homotopy groups are defined recursively as $\pi_n(X, x_0) = \pi_{n-1}(\Omega_{x_0}, C)$ where C is the constant loop at x_0. In other words, the n-th homotopy group is defined as the set of all equivalence classes (under the relationship "homotopic with endpoints held fixed") of loops rooted at C in the loop space Ω_{x_0} of the previous step.

So, for example, $\pi_2(S^1, 1) = \pi_1(\Omega_1, C)$. So, let us examine the loop space around C in $\S^1$. The path-connected region of the loop space of S^1 which contains C is all of the paths homotopic to the constant loop at 1. By a similar argument as the last example of the previous section, this space is the space of all continuous functions from I onto R with $f(0) = f(1) = 1$. So, the loop space of S^1 is roughly $I \times R$. Since any loop in $I \times R$ is collapsible to a point, all loops in Ω_1 of S^1 are homotopic to the constant loop. In other words, $\pi_2(S^1, 1)$ is trivial.

5. Homologies

We're going to shift gears entirely at the moment. The development of homology theory doesn't really depend upon the apparatus developed above for homotopies. But, in section 5.7, we will show a natural connection between homotopies and homologies. My comfort with and understanding of the following is not as deep yet as it is with the preceding. As a result, most of what follows will be more of the upshot of the development of homology as opposed to the full apparatus of development. The notation gets a bit overcrowded in this section, so I will summarize it here:

Δ_q: This is the q-dimensional simplex of Euclidean space. The vectors which make up the basis of this are denoted:

$$E_0 = (0, 0, 0, \ldots, 0, \ldots)$$

$$E_1 = (1, 0, 0, \ldots, 0, \ldots)$$

$$E_2 = (0, 1, 0, \ldots, 0, \ldots)$$

$$\ldots$$

So, for example, q_2 is a triangle with vertices E_0, E_1, and E_2. q_0 is the single point E_0 at the origin.

∂: The boundary operator. It is first described here in section 5.4 on page 16.

$S_q(X)$: The free R-module generated by all of the singular q-simplexes. This is discussed in section 5.2 on page 16.

Z_q: Z_q is a submodule of $S_q(X)$. It is all of the elements of $S_q(X)$ with zero boundary. So, $z \in Z_q$ if and only if z is a singular q-chain and $\partial(z) = 0$. Elements of Z_q are called cycles.

B_q: B_q is a submodule of Z_q. It is all of the cycles which are boundaries of singular $(q+1)$-chains. So, $b \in B_q$ if and only if $b \in Z_q$ and there exists some $\hat{b} \in S_{q+1}(X)$ such that $\partial(\hat{b}) = b$. It is worthy of note that the empty cycle 0 is the boundary of every element in Z_{q+1}, so $0 \in B_q$.

$H_q(X; R)$: The quotient module Z_q/B_q is the q-th singular homology module of X. It can be denoted $H_q(X)$ if the ring R is irrelevant or obvious.

$S(X)$: This is the chain complex formed by the modules $S_q(X)$ and the homomorphisms ∂_q. Chain complexes are described in section 5.6.

5.1. **Singular q-Simplexes.** At the heart of [GH]'s development of homology is the singular q-complex. But, before we get to complexes, we will start with simplexes. A singular q-simplex in a topological space X is a map $\sigma : \Delta_q \to X$. As examples, a singular 0-simplex is a map from E_0 to a point in X. The map $\sigma : \Delta_0 \to I$ given by $\sigma(0) = 1$ is one example of a singular 0-simplex. A singular 1-simplex is a path in X. A singular 2-simplex is a continuous map of the standard geometric triangle Δ_2 into the space X. It is important to realize that the q-simplex is the *map* and not the *image* of the map. For example, the two 1-simplexes $\sigma_0(s) = s$ and $\sigma_1(s) = s^2$ (playing a little loose here with turning vectors between E_0 and E_1 into real numbers by inclusion) are two different simplexes even though they have the same image I.

5.2. **Singular q-Chains.** A singular q-chain is a linear combination of singular q-simplexes in a given space X with coefficients from a given unitary ring R. Usually $R = \mathbf{Z}$, but it could easily be any other unitary ring like $\mathbf{R}$ or $\mathbf{C}$, etc. The free R-module generated by all of the singular q-simplexes is denoted $S_q(X)$. An example of a member of $S_0(I)$ is the singular q-chain $3\sigma + 2\tau - 10\rho$ where σ maps E_0 to 0, τ maps E_0 to 1 and ρ maps E_0 to $\frac{1}{2}$.

5.3. **Faces.** The i-th face of Δ_q is defined in terms of a function F_q^i which maps $\Delta_{q-1} \to \Delta_q$. This function is the unique affine map from Δ_{q-1} onto the ordered set of points $(E_0, E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_q)$. So, F_q^i maps Δ_{q-1} homeomorphically (and preserving straight lines) onto the set-theoretic face of Δ_q that is opposite E_i.

The i-th face of a singular q-simplex σ in a space X is denoted $\sigma^{(i)}$ and is the singular $(q-1)$-simplex $\sigma \circ F_q^i$. Again, do not forget that this is a map, not the image of a map.

5.4. **Boundaries.** The boundary of a singular q-simplex is the singular $(q-1)$-chain (member of $S_{q-1}(X)$) defined by

$$\partial(\sigma) = \sum_{i=0}^{q} (-1)^i \sigma^{(i)} \tag{2}$$

And, because S_q is a free R-module, we can extend this boundary operator into an R-module homomorphism $S_q(X) \to S_{q-1}(X)$ naturally as

$$\partial(\sum \nu_\sigma \sigma) = \sum \nu_\sigma \partial(\sigma)$$

That is to say that a boundary of a singular q-chain is the singular q-chain (with the same coefficients) of the boundaries of the singular q-simplexes.

An interesting effect of this boundary operator is that the boundary of a boundary is always empty. In other words, $\partial \circ \partial$ is the zero map. The proof of this is a notational nightmare. But, it hinges on the fact that $F_q^i F_{q-1}^j = F_q^j F_{q-1}^{i-1}$ for $j < i$. What this is saying is that the j-th face of the i-th face of a simplex is the same as the $(i-1)$-th face of the j-th simplex. If you consider the triangle with points E_0, E_1, and E_2, then you will see that the 1st face is the segment from E_0 to E_2. The 2nd face is the segment between E_0 and E_1. So, for $i = 2$ and $j = 1$, the equation goes:

$$\begin{aligned} F_2^2 F_1^1 &= F_2^1 F_1^1 \\ [E_0, E_2] F_1^1 &= [E_0, E_1] F_1^1 \\ E_0 &= E_0 \end{aligned}$$

With the definition given above in equation 2, these two instances of the same 0-simplex end up with opposite signs. Thus, they cancel each other out.

5.5. **The Homology Groups.** All of the elements of $S_q(X)$ which have a zero boundary form a submodule Z_q of $S_q(X)$. So, $z \in Z_q$ if and only if z is a singular q-chain and $\partial(z) = 0$. Elements of Z_q are called cycles.

All of the elements of Z_q which are the boundaries of a $(q+1)$-simplex form a submodule of Z_q called B_q. It is all of the cycles which are boundaries of singular $(q+1)$-chains. So, $b \in B_q$ if and only if $b \in Z_q$ and there exists some $\hat{b} \in S_{q+1}(X)$ such that $\partial(\hat{b}) = b$.

The quotient module Z_q/B_q is the q-th singular homology module of X. It is denoted $H_q(X; R)$ or just $H_q(X)$ if the ring R is obvious.

Two elements of Z_q map to the same element of $H_q(X)$ if their formal difference as elements of $S_q(X)$ is in B_q. That is to say that if z_0 and z_1 are cycles and $z_0 - z_1$ is a boundary of a $(q+1)$-simplex, then $[z_0] = [z_1]$. If that is the case, then z_0 and z_1 are said to be homologous.

5.6. **Chain Complexes.** We can abstract all of this boundary stuff a bit further. We can take any sequence $C = \{C_q, \partial_q\}$ of free R-modules C_q and homomorphisms $\partial_q : C_q \to C_{q-1}$. And, if we restrict the homomorphisms so that $\partial_q \circ \partial_{q+1} = 0$ for all q, then we've taken the whole of the notions from singular q-chains and abstracted them to any sequence of homomorphisms of free R-modules where the application of two successive homomorphisms is zero. This is called a chain complex. Put yet another way, it doesn't matter what the C_q or the homomorphisms ∂_q are so long as the image of ∂_{q+1} is the kernel of ∂_q.

The particular chain complex described in the previous section of $S_q(X)$ under the boundary operator homomorphisms ∂_q is often abbreviated $S(X)$.

The q-th homology module of C parallels the last section. It is the quotient module of $Z_q(C)$ (the set of cycles in C_q—those elements z with $\delta_q(z) = 0$) and $B_q(C)$ (the set of cycles in C_q which are the images of elements in C_{q+1} under the homomorphism ∂_{q+1}). So, $H_q(C) = Z_q(C)/B_q(C)$.

It is often useful to define an augmentation ϵ to a chain complex. The augmentation is an onto homomorphism from C_0 to the ring R so that $\epsilon \circ \partial_1 = 0$. This means that the image of ∂_1 is a subset of the kernel of ϵ and that $C_0/\text{kern}\ \epsilon$ is isomorphic to the ring R. This extends the chain the whole way into the ring R.

5.7. **Connection to the Homotopy Groups.** A sequence of homomorphisms $\{f_q\}$ where $f_q : C_q \to \hat{C}_q$ is called a chain map if it induces a chain complex in the $\hat{C}_q$'s. In other words, $\hat{\partial}_q \circ f_q$ has to be the same as $f_{q-1} \circ \partial_q$. Put a bit less formally, the functions f_q map C_q's into $\hat{C}_q$'s. Because of this, the boundary maps on the C_q's along with the f_q's create an implied "boundary map" between the $\hat{C}_q$'s. If this implied map is really a boundary map, that is to say that $\hat{\partial}_q \circ \hat{\partial}_{q+1} = 0$, then the sequence $\{f_q\}$ is a chain map. It homomorphically maps one chain complex to another.

If f and g are homotopic maps from X to Y, then the sequence of maps $S(f)$ and $S(g)$ (which are the natural maps that f and g extend onto the various S_q in the sequence S) are chain homotopic maps from $S(X)$ into $S(Y)$. If f and g are homotopic maps, then for $q \geq 0$, the induced homomorphisms of $H_q(f)$ and $H_q(g)$ on the homology modules are equal.

There is also a natural homomorphism $\chi : \pi_1(X, x_0) \to H_1(X; \mathbf{Z})$ which sends the equivalence class (under the relationship "homotopic with endpoints held fixed") of a loop γ into the equivalence class (under the relationship "is homologous to") of the singular 1-simplex γ. Additionally, if X is path-connected, then χ is onto.

5.8. **Exact Homology Sequences.** If A is a subspace of X, then for $q \geq 0$, $S_q(A)$ is a submodule of $S_q(X)$. It is made up of only those members of $S_q(X)$ which map the standard q-simplex Δ_q into A. We can form the quotient module of $S_q(X)$ and $S_q(A)$ and call it $S_q(X, A)$. This quotient module identifies elements of $S_q(X)$ which differ only by members of $S_q(A)$. The mappings of these $S_q(X)$ into $S_q(X, a)$ forms a chain map from the chain $S(X)$ into the chain $S(X, A)$.

The homology group on $S_q(X, A)$ is defined as before in section 5.5 except with the quotient module of $S_q(X)/S_q(A) = S_q(X, A)$. The cycles in the quotient module form the submodule $Z_q(X, A)$. The boundaries in the $Z_q(X, A)$ module form the submodule $B_q(X, A)$. And, the relative homology group $H_q(X, A)$ (the homology group of A relative to X) is isomorphic to the quotient $Z_q(X, A)/B_q(X, A)$.

There is a natural "connecting homomorphism" that connects $H_q(X, A)$ to $H_{q-1}(A)$. If we have a relative q-cycle $z \in Z_q(X, A)$, then it is in the relative homology class $[z] \in H_q(X, A)$. The boundary of z is a $(q-1)$-cycle on A. The homology class of $\partial(z)$ is in $H_{q-1}(A)$.

Along with the inclusion homomorphism, that maps each member of $H_q(A)$ into $H_q(X)$, we can form a long chain complex for a set and a subset of it. Given $A \subset X$, we can form the chain

$$\ldots \to H_q(A) \to H_q(X) \to H_q(X, A) \to H_{q-1}(A) \to \ldots$$

The composite of any two consecutive homomorphisms in this chain is zero. The part that took me the longest to see was how the inclusion map didn't get in the way of the composition of consecutive homomorphisms being the zero map. But, upon further consideration, a cycle in $S_q(A)$ is a boundary in $S_q(X)$ because the image of a cycle in $S_q(A)$ under the inclusion map is a cycle (which is how boundaries are defined) in $S_q(X)$.

6. Conclusion

I have no major conclusions from this. I am more eager than ever to pursue Milnor's proof of the existence of exotic structures on S^7. And, now I have a whole bunch more ammunition and practice parsing terse texts to fight that battle. Thank you for your help.

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