PLANETARY MOTION IN FOUR DIMENSIONS

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1. Introduction

This follows the development in *Calculus with Analytic Geometry: Second Edition* by John B. Fraleigh (©1985; Addison-Wesley; Reading, Massachussetts; ISBN 0-201-1201-0).

The only independent vector quantities involved in a planetary orbit are the vector from the sun to the planet ($\vec{\mathbf{r}}$) and the velocity of the planet ($\vec{\mathbf{v}}$). All other vector quantities are derived from these and from various scalar quantities (the mass of the sun (M), the mass of the planet (m), etc.). These two vectors determine a plane. Without loss of generality, we can assume this happens in the x-y plane. As we shall see, there is nothing in the orbital mechanics which would cause the planet to move out of that plane.

2. Coordinate System

All of the following derivation will done in a two-dimensional coordinate system. This coordinate system is defined in terms of two perpendicular vectors. The first vector ($\hat{\mathbf{u}}_r$) is a unit vector in the direction from the sun to the planet. The second vector ($\hat{\mathbf{u}}_\theta$) is a unit vector in the orbital plane, perpendicular to the first vector ($\hat{\mathbf{u}}_r$) and pointing in the direction of positive angular motion.

To convert from this coordinate system back to x-y coordinates, one can use the following identities:

$$\begin{aligned} \hat{\mathbf{u}}_r &= (\cos\theta)\hat{\mathbf{i}} + (\sin\theta)\hat{\mathbf{j}} \\ \hat{\mathbf{u}}_\theta &= -(\sin\theta)\hat{\mathbf{i}} + (\cos\theta)\hat{\mathbf{j}} \end{aligned} \tag{2.1}$$

Differentiating these with respect to the angle (θ), we find that:

$$\begin{aligned} \frac{\hat{\mathbf{u}}_r}{d\theta} &= (-\sin\theta)\hat{\mathbf{i}} + (\cos\theta)\hat{\mathbf{j}} = \hat{\mathbf{u}}_\theta \\ \hat{\mathbf{u}}_\theta &= -(\cos\theta)\hat{\mathbf{i}} - (\sin\theta)\hat{\mathbf{j}} = -\hat{\mathbf{u}}_r \end{aligned} \tag{2.2}$$

3. Position, Velocity, and Acceleration

The position ($\vec{\mathbf{r}}$) can be expressed in terms of its direction ($\hat{\mathbf{u}}_r$) and its magnitude (r) as follows:

$$\vec{\mathbf{r}} = r\hat{\mathbf{u}}_r \tag{3.1}$$

The velocity ($\vec{\mathbf{v}}$) is the time-derivative of the position:

$$\begin{aligned} \vec{\mathbf{v}} &= \frac{d\vec{\mathbf{r}}}{dt} \\ &= r\frac{d\hat{\mathbf{u}}_r}{dt} + \dot{r}\hat{\mathbf{u}}_r \\ &= r\dot{\theta}\frac{d\hat{\mathbf{u}}_r}{d\theta} + \dot{r}\hat{\mathbf{u}}_r \\ &= r\dot{\theta}\hat{\mathbf{u}}_\theta + \dot{r}\hat{\mathbf{u}}_r \\ &= \dot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\hat{\mathbf{u}}_\theta \end{aligned} \tag{3.2}$$

The acceleration ($\vec{\mathbf{a}}$) is the time-derivative of the velocity:

$$\begin{aligned} \vec{\mathbf{a}} &= \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2} \\ &= \dot{r}\frac{d\hat{\mathbf{u}}_r}{dt} + \ddot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\frac{d\hat{\mathbf{u}}_\theta}{dt} + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\mathbf{u}}_\theta \\ &= \dot{r}\dot{\theta}\frac{d\hat{\mathbf{u}}_r}{d\theta} + \ddot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\frac{d\theta}{dt}\frac{d\hat{\mathbf{u}}_\theta}{d\theta} + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\mathbf{u}}_\theta \\ &= \dot{r}\dot{\theta}\hat{\mathbf{u}}_\theta + \ddot{r}\hat{\mathbf{u}}_r + r\dot{\theta}^2(-\hat{\mathbf{u}}_r) + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\mathbf{u}}_\theta \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{u}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\mathbf{u}}_\theta \end{aligned} \tag{3.3}$$

4. Force Acting on the Planet

The only force acting on the planet is the force of gravity of the sun. According to Newton's second law of motion, the acceleration ($\vec{\mathbf{a}}$) multiplied by the mass (m) is equal to the force ($\vec{\mathbf{F}}$) causing the acceleration:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \tag{4.1}$$

That force is the only one acting upon the planet. Further, that force always acts radially. Thus, there is no acceleration in the direction ($\hat{\mathbf{u}}_\theta$) which is perpendicular to the radius. So, that component in equation 3.3 must be zero.

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{4.2}$$

One thing this means for us, which we will use later, is that the time-based derivative of the square of the radius multiplied derivative of the angle is zero.

$$\begin{aligned} \frac{d}{dt}\left(r^2\dot{\theta}\right) &= r^2\ddot{\theta} + 2r\dot{r} \\ &= r\left(r\ddot{\theta} + 2\dot{r}\right) \\ &= r0 = 0 \end{aligned} \tag{4.3}$$

Which in turn means that the square of the radius multiplied by the derivative of the angle is a constant.

$$\begin{aligned} \frac{d}{dt}\left(r^2\dot{\theta}\right) &= 0 \\ r^2\dot{\theta} &= K \end{aligned} \tag{4.4}$$

This K is the angular momentum L of the planet divided by its mass.

The magnitude of the radial force is determined using Newton's universal law of gravitation which relates the mass of the planet (m), the mass of the sun (M), the universal gravitational constant (G), and the distance (r) from the planet to the sun. Here, we have modified Newton's law of gravity by making it an inverse cube law so that it makes more sense in four dimensions.

$$F = -\frac{mMG}{r^3} \tag{4.5}$$

The force of gravity is responsible for all of the acceleration in the radial direction ($\hat{\mathbf{u}}_r$).

$$\begin{aligned} \vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{u}}_r &= -\frac{mMG}{r^3}\hat{\mathbf{u}}_r \\ \ddot{r} - r\dot{\theta}^2 &= -\frac{MG}{r^3} \\ &= -\alpha\frac{1}{r^3}, \quad \alpha = MG \end{aligned} \tag{4.6}$$

5. Radius as a Function of Angle

Squaring equation 4.4 and dividing both sides by the cube of the radius, we see that:

$$\frac{K^2}{r^3} = \frac{\left(r^2\dot{\theta}\right)^2}{r^3} = r\dot{\theta}^2 \tag{5.1}$$

Substituting that into equation 4.6, we obtain:

$$\begin{aligned} \ddot{r} - \frac{K^2}{r^3} &= -\alpha\frac{1}{r^3} \\ \ddot{r} &= \frac{1}{r^3}\left(K^2 - \alpha\right) \end{aligned} \tag{5.2}$$

Multiplying both sides of that equation by the time-based derivative of the radius, one can then integrate both sides.

$$\begin{aligned} \dot{r}\ddot{r} &= \frac{\dot{r}}{r^3}\left(K^2 - \alpha\right) \\ \frac{1}{2}\dot{r}^2 &= -\frac{1}{2r^2}\left(K^2 - \alpha\right) + \frac{1}{2}C\prime \\ \dot{r}^2 &= \frac{1}{r^2}\left(\alpha - K^2\right) + C\prime \end{aligned} \tag{5.3}$$

Now, we will perform the substitution $p = \frac{1}{r}$. With that, we find that $\dot{r} = -\frac{\dot{p}}{p^2}$. This gives us the equation:

$$\frac{\dot{p}^2}{p^4} = p^2\left(\alpha - K^2\right) + C\prime \tag{5.4}$$

We are going to massage the left-hand side with the help of equation 4.4. From equation 4.4, we know that $\frac{\dot{\theta}}{p^2} = K$

$$\begin{aligned} \frac{\dot{p}^2}{p^4} &= \frac{1}{p^4} (\dot{p})^2 \\ &= \frac{1}{p^4} \left(\frac{dp}{dt} \right)^2 \\ &= \frac{1}{p^4} \left(\frac{d\theta}{dt} \right)^2 \left(\frac{dp}{d\theta} \right)^2 \\ &= K^2 \left(\frac{dp}{d\theta} \right)^2 \end{aligned} \tag{5.5}$$

Using that in equation 5.4, we get:

$$\begin{aligned} \frac{\dot{p}^2}{p^4} &= p^2 \left(\alpha - K^2 \right) + C\prime \\ K^2 \left(\frac{dp}{d\theta} \right)^2 &= p^2 \left(\alpha - K^2 \right) + C\prime \\ \left(\frac{dp}{d\theta} \right)^2 &= p^2 \left(\frac{\alpha}{K^2} - 1 \right) + C \end{aligned} \tag{5.6}$$

Taking the derivative of both sides with respect to theta, we find that:

$$\begin{aligned} 2 \left(\frac{dp}{d\theta} \right) \left(\frac{d^2p}{d\theta^2} \right) &= 2p \left(\frac{\alpha}{K^2} - 1 \right) \frac{dp}{d\theta} \\ \left(\frac{dp}{d\theta} \right) \left[\left(\frac{d^2p}{d\theta^2} \right) - \beta p \right] &= 0, \quad \beta = \left(\frac{\alpha}{K^2} - 1 \right) \end{aligned} \tag{5.7}$$

This means that either:

$$\left(\frac{dp}{d\theta} \right) = 0 \quad \text{or} \quad \left(\frac{d^2p}{d\theta^2} \right) - \beta p = 0 \tag{5.8}$$

6. Shapes of "Orbits"

If $\frac{dp}{d\theta}$ is zero, then p is constant, regardless of the angle. Since p is the inverse of the radius, when p is constant so is the radius. Thus, the orbit is a circle.

If, on the other hand, $\frac{d^2p}{d\theta^2} - \beta p$ is zero, then we are left with the differential equation:

$$\frac{d^2p}{d\theta^2} = \beta p \tag{6.1}$$

This can be broken up into three case depending on whether the constant β is less than, equal to, or greater than zero.

First, however, we will think a bit about the constant β. From equations 5.7 and 4.6, we know that $\beta = \left(\frac{GM}{K^2} - 1 \right)$. The constant K is proportional to the angular momentum of the planet. For β to be zero, the angular momentum must perfectly balance the force of gravity imposed by the sun. If the force of gravity is smaller, then β will be negative. If the force of gravity is larger, then β will be positive.

In the case where β equals zero, the solution to equation 6.1 is:

$$p(\theta) = v\theta + \frac{1}{r_0}$$
$$\frac{1}{r(\theta)} = v\theta + \frac{1}{r_0} \tag{6.2}$$
$$r(\theta) = \frac{1}{v\theta + \frac{1}{r_0}}$$

for constants v and r_0. If v is zero, then we again have a circular orbit of radius r_0. If v is non-zero, then the orbit is either spiralling inward or outward as v is positive or negative, respectively. This is a spiral which will either always approach the sun or always get further from the sun. This jives with our understanding of the constant β. When β is zero, the planet's distance from the sun maintains a steady rate of change with respect to its angle from the sun because the planet's angular momentum is perfectly balancing the sun's gravitational force.

In the case where β is greater than zero, the solution to equations 6.1 is:

$$p(\theta) = \frac{1}{r_0} e^{\theta\sqrt{\beta}} \tag{6.3}$$
$$r(\theta) = r_0 e^{-\theta\sqrt{\beta}}$$

This, too, is a spiralling sort of shape. But, it is one that inevitably crashes into the sun. This also jives with our understanding of the constant β. In this case, the gravitational attraction of the sun overpowers the momentum of the planet.

In the case where β is less than zero, the solution to equations 6.1 is:

$$p(\theta) = \frac{1}{r_{\min}} \cos\left(\theta\sqrt{-\beta} + \theta_0\right) \tag{6.4}$$
$$r(\theta) = \frac{1}{\frac{1}{r_{\min}} \cos\left(\theta\sqrt{-\beta} + \theta_0\right)}$$

Again, this is a spiralling sort of shape. But, it is one that will never approach closer to the sun than a radius of $r_{\min}$. The orbit spirals in from infinity, gets as close as $r_{\min}$, and spirals back out toward infinity. This also jives with our understanding of the constant β. In this case, the momentum of the planet overpowers the gravitational attraction of the sun.