# RESTRICTIONS OF NON-COMMUTATIVE ALGEBRAS WHICH INDUCE COMMUTATIVITY

PATRICK FLECKENSTEIN

ABSTRACT. The non-trivial Clifford algebras over the ring $\mathbb{Z}$ are non-commutative. On the other hand, the Clifford algebras over the ring $\mathbb{Z}_2$ are commutative.

This result is non-trivial in that it does not depend solely on either $\mathbb{Z}_2$ or the Clifford algebras. The non-trivial Clifford algebras over the ring $\mathbb{Z}_3$ and the matrix algebra with basis $\{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$ over the ring $\mathbb{Z}_2$ are still non-commutative.

In this paper, we analyze the structure of the Clifford algebras over the ring $\mathbb{Z}_2$, explore what relationship between these algebras and the ring $\mathbb{Z}_2$ induces the commutativity, and lay out some of the necessary and/or sufficient conditions for such a shift in commutativity.

## 1. Motivation

In non-commutative rings there are some elements which commute with every other element. The set of elements of the ring $\mathbf{R}$ which commute with every other element of $\mathbf{R}$ is called the center of $\mathbf{R}$ and is denoted $\mathrm{Cen}(\mathbf{R})$. In the form of equations,

$$c \cdot a = a \cdot c \qquad \forall c \in \mathrm{Cen}(\mathbf{R}), \forall a \in \mathbf{R} \tag{1.1}$$

Because $0 \in \mathrm{Cen}(\mathbf{R})$, $\mathrm{Cen}(\mathbf{R})$ can never be empty. And, in non-commutative rings, $\mathrm{Cen}(\mathbf{R}) \neq \mathbf{R}$. It is then natural to formulate the question, "How non-commutative is a ring?". One way to approach that question is to compare the cardinality of $\mathrm{Cen}(\mathbf{R})$ and the cardinality of $\mathbf{R}$. Unfortunately, the answers to that question will not be very interesting unless the ring is finite.

Another way to answer the question "How non-commutative is a ring?" is to compare the number of generators of $\mathrm{Cen}(\mathbf{R})$ with the number of generators of $\mathbf{R}$. This approach has potential if $\mathbf{R}$ is finitely generated. But, this paper does not pursue this.

Another way to approach the question is to rephrase the question as "What are the minimum requirements that a ring homomorphism $\phi$ must satisfy so that $\phi(\mathbf{R})$ is commutative?" A special case of this question is "Given a non-commutative algebra, over what rings is it commutative?" This paper will tackle that question as it relates to the Clifford algebras.

Clifford algebras were not my first choice of targets. I had originally been exploring the quaternion algebra. The quaternion algebra, however, is one

*Date*: 2000-05-05.

subalgebra of the three-vectored Clifford algebra $\mathcal{C}\ell_3$. As the work of this paper holds for all Clifford algebras and their subalgebras, the scope of this paper was broadened to encompass them.

## 2. Algebras in General

An algebra $\mathcal{A}$ is a vector space ${}_{\mathbf{F}}\mathbf{V}$ with vectors from $\mathbf{V}$ and scalars from a field $\mathbf{F}$ and a bilinear product defined which maps ${}_{\mathbf{F}}\mathbf{V} \times {}_{\mathbf{F}}\mathbf{V} \to {}_{\mathbf{F}}\mathbf{V}$.

In this paper, we will be more concerned with algebras over rings, than algebras over fields. Our scalars will come from a ring $\mathbf{R}$ instead of a field $\mathbf{F}$. Nothing about the product requires the use of a field instead of a ring. A vector space with scalars from a ring $\mathbf{R}$ is called an $\mathbf{R}$-module. There does not appear to be a term for an algebra with scalars from a ring. In this paper, we will use the term *algebra* to mean an algebra over a ring.

So, for the purposes of this paper, an algebra $\mathcal{A}$ is an $\mathbf{R}$-module ${}_{\mathbf{R}}\mathbf{V}$ with vectors from $\mathbf{V}$ and scalars from a ring $\mathbf{R}$ and a bilinear product defined which maps ${}_{\mathbf{F}}\mathbf{V} \times {}_{\mathbf{F}}\mathbf{V} \to {}_{\mathbf{F}}\mathbf{V}$. The fact that the product is bilinear means that for vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ and scalars $\alpha, \beta \in \mathbf{R}$, the product obeys:

$$(\alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v}) * \mathbf{w} = \alpha \cdot (\mathbf{u} * \mathbf{w}) + \beta \cdot (\mathbf{v} * \mathbf{w}) \tag{2.2}$$

$$\mathbf{u} * (\alpha \cdot \mathbf{v} + \beta \cdot \mathbf{w}) = \alpha \cdot (\mathbf{u} * \mathbf{v}) + \beta \cdot (\mathbf{u} * \mathbf{w}) \tag{2.3}$$

A normed vector space[1] is a vector space in which every vector $\mathbf{a}$ has a magnitude $\|\mathbf{a}\|$ such that $\|\mathbf{a}\| \geq 0$, $\|\mathbf{a}\| = 0$ iff $\mathbf{a} = \mathbf{0}$, for any scalar $k$ we have $\|k \cdot \mathbf{a}\| = |k| \, \|\mathbf{a}\|$, and $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. The most common norm is the $L2$-Norm which is

$$\|\mathbf{a}\|_2 = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \tag{2.4}$$

## 3. The Cross-Product

For three-dimensional vectors, the vector cross-product is a bilinear product which maps ${}_{\mathbf{F}}\mathbf{V} \times {}_{\mathbf{F}}\mathbf{V} \to {}_{\mathbf{F}}\mathbf{V}$. As such, the real-valued three-dimensional vectors form an algebra. But, this is an accident of three dimensions.

The cross-product maps a pair of vectors to a perpendicular vector whose length is the area of the parallelogram with the two original vectors as sides. In two-dimensions, there just isn't a vector perpendicular to two non-parallel vectors. And, in attempting to scale this up in dimensions, one is looking for a vector product that satisfies[2]:

$$(\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k) \cdot \mathbf{a}_i \qquad \forall 1 \leq i \leq k \tag{3.5}$$

$$|(\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k)| = |\mathcal{V}| \tag{3.6}$$

[1] [Weisstein] Norm

[2] [Lounesto] p. 98

where $\mathcal{V}$ is the signed-volume of the parallelpiped determined by the vectors. The only non-trivial solutions[3] to this are:

$$\begin{array}{llll} n & \text{dimensions with} & n-1 & \text{factors} \\ 7 & \text{dimensions with} & 2 & \text{factors} \\ 8 & \text{dimensions with} & 3 & \text{factors} \end{array} \tag{3.7}$$

So, in order to form a class of algebras which will be useful for vectors of any given dimension, one must abandon the cross-product. A different product, called the exterior product or outer product, can be used in place of the cross-product.

The exterior product of $n+1$ vectors from $\mathbb{R}^n$ is $\mathbf{0}$. The exterior product of $n$ linearly independent vectors from $\mathbb{R}^n$ has the same magnitude as the $n$-factored cross-product in $\mathbb{R}^{n+1}$ would have. However, the exterior product is useful for other than $n$ terms.

The 3-dimensional cross-product maps $\mathbf{a}\times\mathbf{b}$ to a vector whose magnitude is $|\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin\theta$ where $\theta$ is the angle between the vectors and $|\mathbf{x}|$ is the $L2$-norm defined in equation 2.4. The exterior product in $n$-dimensions has this same magnitude, but the result is not a vector. The result is called a bivector[4] or 2-form[5]. And,

$$|\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin\theta$$

Similarly, the exterior product of 3 vectors is a trivector or a 3-form.

The exterior product, like the cross-product, is antisymmetric. That is that:

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a} \tag{3.8}$$

The exterior product is a bilinear product which maps $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$. The algebra of the exterior product is called the Grassman algebra. The Clifford algebras which follow are a related to the Grassman algebras but have some other properties which make them more useful in $n$-dimensional geometry.

## 4. The Clifford Algebras

To form the $n$-vectored Clifford algebra $\mathcal{C}\ell_n$, one starts with $n$ orthogonal unit vectors $\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n\}$ from a normed vector space and creates an algebra so that for all $\mathbf{r}$ we have $\mathbf{r}^2=\mathbf{r}\mathbf{r}=|\mathbf{r}|^2$.

*Note.* The Clifford product of $\mathbf{a}$ and $\mathbf{b}$ is written $\mathbf{ab}$ to distinguish it from the other familiar vector products $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\wedge\mathbf{b}$.

There is one other notational convention that is important for the Clifford algebras. The product $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_n}$ is usually abbreviated $\mathbf{e}_{i_1 i_2\cdots i_n}$. Because the product of vectors is associative, this doesn't create any ambiguity.

Now, let us illustrate the Clifford algebras with an example. Consider a vector in ${}_{\mathbf{R}}\mathcal{C}\ell_2$. It would be $x\mathbf{e}_1+y\mathbf{e}_2$ where $x,y\in\mathbf{R}$. In order to satisfy the

[3] [Lounesto] p. 98
[4] [Gull] A Little Un-Learning
[5] [Weisstein] Wedge Product

above requirements, $(x\mathbf{e}_1 + y\mathbf{e}_2)^2$ must equal $x^2 + y^2$. Simply multiplying out the right hand side, we have

$$x^2{\mathbf{e}_1}^2 + xy\mathbf{e}_{12} + yx\mathbf{e}_{21} + y^2{\mathbf{e}_2}^2 = x^2 + y^2 \tag{4.9}$$

This means that ${\mathbf{e}_1}^2 = {\mathbf{e}_2}^2 = 1$ and $xy\mathbf{e}_{12} = -yx\mathbf{e}_{21}$. It is easy to see from this that if either $x$ or $y$ is a non-zero member of Cen($\mathbf{R}$) and $x \cdot y \neq 0$, then $\mathbf{e}_{12} = -\mathbf{e}_{21}$. In order to make them useful over all rings, we define the product this way $\mathbf{e}_{12} = -\mathbf{e}_{21}$.

This pattern holds for all of the Clifford algebras. If $\mathbf{e}_i$ and $\mathbf{e}_j$ are two of the $n$ orthogonal unit vectors, then

$$\mathbf{e}_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\mathbf{e}_{ji} & \text{if } i \neq j. \end{cases} \tag{4.10}$$

Now, if we use this result to multiply two vectors in $\mathbf{R}^2$, we have

$$\begin{aligned} (a\mathbf{e}_1 + b\mathbf{e}_2) \cdot (c\mathbf{e}_1 + d\mathbf{e}_2) &= ac{\mathbf{e}_1}^2 + ad\mathbf{e}_{12} + bc\mathbf{e}_{21} + bd{\mathbf{e}_2}^2 \\ &= (ac + bd) + ad\mathbf{e}_{12} - bc\mathbf{e}_{12} \\ &= (ac + bd) + (ad - bc)\mathbf{e}_{12} \end{aligned} \tag{4.11}$$

This looks like an odd result. We have multiplied two vectors and gotten something that is a scalar and whatever this $\mathbf{e}_{12}$ thing happens to be.

We can see that $\mathbf{e}_{12}$ cannot be either a scalar or a vector because

$$\begin{aligned} {\mathbf{e}_{12}}^2 &= \mathbf{e}_{12}\mathbf{e}_{12} \\ &= -\mathbf{e}_{21}\mathbf{e}_{12} \\ &= -\mathbf{e}_2{\mathbf{e}_1}^2\mathbf{e}_2 \\ &= -\mathbf{e}_2\mathbf{e}_2 \\ &= -1 \end{aligned} \tag{4.12}$$

This element $\mathbf{e}_{12}$ is called a bivector or a 2-form. It happens that because $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal, that $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$.

If we add 1 and $\mathbf{e}_{12}$ into the algebra, we can achieve closure. It is easy to see that multiplying by scalars does not go outside of the algebra. And, we can see that multiplying a vector and bivector results in a vector because:

$$\begin{aligned} \mathbf{e}_1\mathbf{e}_{12} &= {\mathbf{e}_1}^2\mathbf{e}_2 \\ &= \mathbf{e}_2 \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathbf{e}_{12}\mathbf{e}_1 &= -\mathbf{e}_{21}\mathbf{e}_1 \\ &= -\mathbf{e}_2{\mathbf{e}_1}^2 \\ &= -\mathbf{e}_2 \end{aligned} \tag{4.14}$$

And, by similar reasoning

$$\mathbf{e}_2\mathbf{e}_{12} = -\mathbf{e}_1 \tag{4.15}$$

$$\mathbf{e}_{12}\mathbf{e}_2 = \mathbf{e}_1 \tag{4.16}$$

Thus, a typical element of ${}_{\mathbf{R}}\mathcal{C}\ell_2$ looks like $a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_{12}$ where $a, b, c, d \in \mathbf{R}$. The shorthand notation for this is

$$\mathcal{C}\ell_2 = \mathbf{R} \oplus \mathbf{R}^2 \oplus \bigwedge^2 \mathbf{R}^2 \tag{4.17}$$

This means that an element of $\mathcal{C}\ell_2$ can be expressed as the sum of a scalar, a 2-vector, and a bivector. The notation $\bigwedge^2 \mathbf{R}^2$ here means the exterior product of two orthogonal vectors $\mathbf{e}_{ij}$ where $i < j$.

For larger Clifford algebras (ones with more orthogonal unit vectors), one has to incorporate things other than scalars and bivectors. The concepts above all extend naturally. And, as an analogue to equation 4.17, we have

$$\mathcal{C}\ell_n = \mathbf{R} \oplus \mathbf{R}^n \oplus \bigwedge^2 \mathbf{R}^n \oplus \bigwedge^3 \mathbf{R}^n \oplus \cdots \oplus \bigwedge^n R^n \tag{4.18}$$

The notation $\bigwedge^k \mathbf{R}^n$ here means the exterior product of $k$ orthogonal vectors $\mathbf{e}_{i_1 i_2 \ldots i_k}$ where $i_j < i_{j+1}$. Using the convention that $\bigwedge^0 \mathbf{R}^n = \mathbf{R}$, we could express this equation as:

$$\mathcal{C}\ell_n = \bigoplus_{k=0}^{n} \bigwedge^k \mathbf{R}^n \tag{4.19}$$

And, the elements of the $k$-th portion of the direct sum are called $k$-forms.

The portions of equation 4.19 where $k$ is even is denoted $\mathcal{C}\ell_n^+$. We can see that $\mathcal{C}\ell_n^+$ is a subalgebra of $\mathcal{C}\ell_n$. Certainly, $\mathcal{C}\ell_n^+$ is closed under addition. When multiplying a $k$-form by a $j$-form, the resulting product can be reduced by annihilating like subscripts as we did in equations 4.12 through 4.16. The annihilation always happens in pairs. Thus, if both $k$ and $j$ are even-forms, then their product will be an even-form.

4.1. **The Complex.** To employ some of the formalisms above and as a stepping stone to the next subsection, we will show that ${}_{\mathbf{R}}\mathcal{C}\ell_2^+ \cong {}_{\mathbf{R}}\mathbb{C}$.

Consider elements $(a + b\mathbf{e}_{12}), (c + d\mathbf{e}_{12}) \in {}_{\mathbf{R}}\mathcal{C}\ell_2$.

$$(a + b\mathbf{e}_{12}) + (c + d\mathbf{e}_{12}) = (a + c) + (b + d)\,\mathbf{e}_{12} \tag{4.20}$$

$$\begin{aligned}(a + b\mathbf{e}_{12}) \cdot (c + d\mathbf{e}_{12}) &= (ac) + (ad)\,\mathbf{e}_{12} + (bc)\,\mathbf{e}_{12} + (bd)\,{\mathbf{e}_{12}}^2 \\ &= (ac) + (ad + bc)\,\mathbf{e}_{12} - (bd) \\ &= (ac - bd) + (ad + bc)\,\mathbf{e}_{12}\end{aligned} \tag{4.21}$$

We can see that substituting $i$ in for $\mathbf{e}_{12}$ satisfies all of these equations as well.

4.2. **The Quaternions.** In a similar way, the quaternions $\mathbb{H}$ are isomorphic to the algebra $\mathcal{C}\ell_3^+$. The basis elements of $\mathcal{C}\ell_3^+$ are $\{1, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}\}$. With the mappings

$$\begin{aligned} 1 &\leftrightarrow 1 \\ i &\leftrightarrow \mathbf{e}_{12} \\ j &\leftrightarrow \mathbf{e}_{23} \\ k &\leftrightarrow \mathbf{e}_{13} \end{aligned} \tag{4.22}$$

we can easily verify that the quaternion properties hold:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{4.23}$$

Because of this isomorphism, any way in which commutativity can be induced in $\mathcal{C}\ell_n$ (where $n \geq 3$) is directly applicable to the quaternions. If the algebra ${}_{\mathbf{R}}\mathcal{C}\ell_n$ is commutative, every subalgebra will be commutative.

## 5. Inducing Commutativity

The Clifford algebras (except for the trivial $\mathcal{C}\ell_1$) are generally non-commutative. This is a natural consequence of the fact that $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$. However, the structure of the Clifford algebras is such that over some rings, the algebra over the ring will be commutative. The relationship $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$ gives us a good hint about where to start looking for such rings.

### 5.1. **Trivial Rings.**

**Theorem 5.1.** *A Clifford algebra over a ring whose multiplication is identical to zero is commutative.*

*Proof.* In multiplying a term $\alpha\mathbf{e}_{a_1a_2\ldots a_s}$ by a term $\beta\mathbf{e}_{b_1b_2\ldots b_t}$ where $\alpha, \beta \in \mathbf{R}$ and $\mathbf{e}_{a_1a_2\ldots a_s}, \mathbf{e}_{b_1b_2\ldots b_t} \in \mathcal{C}\ell_n$, we will necessarily get $(\alpha\beta)\mathbf{e}_{a_1a_2\ldots a_sb_1b_2\ldots b_t}$ which is $0\mathbf{e}_{a_1a_2\ldots a_sb_1b_2\ldots b_t}$. Adding up any number of these terms will give 0.

Similarly, if we multiply in the other order, we obtain 0. So, $\mathbf{ab} = 0 = \mathbf{ba}$ for all $\mathbf{a}, \mathbf{b} \in {}_{\mathbf{R}}\mathcal{C}\ell_n$. □

One example of such a ring is the subring $\mathbf{R} = \{\bar{0}, \bar{2}\}$ of $\mathbb{Z}_4$. Other examples are easy to construct from any Abelian group $(\mathbf{G}, +)$ by constructing the ring $\mathbf{R} = (\mathbf{G}, +, \cdot_0)$ where $a \cdot_0 b = 0$ for all $a, b \in \mathbf{R}$.

These rings are rather unrewarding though because each simply turns the Clifford algebra $\mathcal{C}\ell_n$ into an algebra where every product is zero.

### 5.2. **Over Positive Rings.**

**Definition 5.1.** Define positive ring to be a ring $\mathbf{R}$ where $x = -x$ for each element $x \in \mathbf{R}$.

**Theorem 5.2.** *A Clifford algebra over a positive commutative ring is commutative.*

*Proof.* Consider the products $\mathbf{ab}$ and $\mathbf{ba}$ where $\mathbf{a} = \alpha\mathbf{e}_{a_1a_2\ldots a_s}$ and $\mathbf{b} = \beta\mathbf{e}_{b_1b_2\ldots b_t}$. Here, $\alpha, \beta \in \mathbf{R}$ with $\mathbf{R}$ a positive commutative ring.

$$\mathbf{ab} = (\alpha \mathbf{e}_{a_1 a_2 \ldots a_s})(\beta \mathbf{e}_{b_1 b_2 \ldots b_t}) = (\alpha\beta)\mathbf{e}_{a_1 a_2 \ldots a_s b_1 b_2 \ldots b_t} \tag{5.24}$$

$$\mathbf{ba} = (\beta \mathbf{e}_{b_1 b_2 \ldots b_t})(\alpha \mathbf{e}_{a_1 a_2 \ldots a_s}) = (\beta\alpha)\mathbf{e}_{b_1 b_2 \ldots b_t a_1 a_2 \ldots a_s} \tag{5.25}$$

Because $\mathbf{R}$ is commutative, we could rewrite equation 5.25 as

$$\mathbf{ba} = (\beta \mathbf{e}_{b_1 b_2 \ldots b_t})(\alpha \mathbf{e}_{a_1 a_2 \ldots a_s}) = (\alpha\beta)\mathbf{e}_{b_1 b_2 \ldots b_t a_1 a_2 \ldots a_s} \tag{5.26}$$

If any of the subscripts $a_i$ are equal to any of the subscripts $b_j$, then we can move them together via the same transpositions of adjacent subscripts as we did in equations 4.12 through 4.16 using the fact that if $c_i \neq c_{i+1}$, then

$$\mathbf{e}_{c_1 c_2 \ldots c_i c_{i+1} \ldots c_{s+t}} = -\mathbf{e}_{c_1 c_2 \ldots c_{i+1} c_i \ldots c_{s+t}} \tag{5.27}$$

If $c_i = c_{i+1}$, then the subscripts annihilate one another because $\mathbf{e}_i \mathbf{e}_i = 1$, so

$$\mathbf{e}_{c_1 c_2 \ldots c_{i-1} c_i c_{i+1} c_{i+2} \ldots c_{s+t}} = \mathbf{e}_{c_1 c_2 \ldots c_{i-1} c_{i+2} \ldots c_{s+t}} \tag{5.28}$$

This process can be applied to reduce the number of subscripts in $\mathbf{ab}$ and $\mathbf{ba}$ to a unique list. This process can also be used to sort the list of subscripts so that in $\mathbf{e}_{c_1 c_2 \ldots c_u}$ we can have $c_i < c_{i+1}$ for $1 \leq i < u$.

If we do this for both $\mathbf{ab}$ and $\mathbf{ba}$, they will both have the same subscript lists. However, it may take us a different number of transpositions to accomplish this for $\mathbf{ab}$ than for $\mathbf{ba}$. Each of these transpositions toggles the sign. So, $\mathbf{ab}$ may differ in sign from $\mathbf{ba}$.

Thus, if $\alpha\beta = -\alpha\beta$, which is true in $\mathbf{R}$, then $\mathbf{ab} = \mathbf{ba}$. Because the product of arbitrary elements of the Clifford algebra are composed of sums of products typified by $\mathbf{ab}$, it follows that the Clifford algebras are commutative over $\mathbf{R}$. □

**Corollary 5.3.** *A Clifford algebra over a positive ring with multiplication identical to zero is commutative.*

*Proof.* This follows trivially from theorem 5.1. We could also prove it by applying theorem 5.2 with the observation that $\alpha\beta = 0 = \beta\alpha$ for all $\alpha, \beta \in \mathbf{R}$. □

The subring $\mathbf{R} = \{\bar{0}, \bar{2}\}$ of $\mathbb{Z}_4$ mentioned earlier is an example of a positive ring with multiplication identical to zero.

**Corollary 5.4.** *A Clifford algebra over a Boolean ring is commutative.*

*Proof.* We can see that all Boolean rings are positive and commutative because by definition, a ring $\mathbf{R}$ is Boolean iff $b^2 = b$ for all $b \in \mathbf{R}$.

From this, we can see that $b = -b$ for all $b \in \mathbf{R}$. Consider $(x + x) \in \mathbf{R}$.

$$\begin{aligned}
(x+x)^2 &= x+x \\
(x+x)(x+x) &= x+x \\
x^2+x^2+x^2+x^2 &= x+x \\
x+x+x+x &= x+x \\
x+x &= 0 \\
x &= -x
\end{aligned} \tag{5.29}$$

Now, using the above and considering for a moment $(x + y) \in \mathbf{R}$, we can see that:

$$\begin{aligned}
(x+y)^2 &= x+y \\
(x+y)(x+y) &= x+y \\
x^2+xy+yx+y^2 &= x+y \\
x+xy+yx+y &= x+y \\
xy+yx &= 0 \\
xy &= -yx \\
xy &= yx
\end{aligned} \tag{5.30}$$

So, all Boolean rings are commutative and positive. Then, by theorem 5.2, this corollary holds. □

The ring $\mathbf{R} = \mathbb{Z}_2$ is an example of a Boolean ring. Some other examples of Boolean rings are $\mathbf{R} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $\mathbf{R} = (\mathcal{P}(\mathbf{X}), \triangle, \cap)$.

*Question.* Is it possible to have a commutative, positive ring that is not trivial or a Boolean ring?

This question plagued me for several days. I could not prove or disprove the existence of such rings. The proof that Boolean rings are positive is straightforward. The question of whether all commutative, positive rings were trivial or Boolean was elusive. However, as we shall see, examples of such rings were right under my nose.

The first non-trivial, commutative, positive ring which I discovered was $\mathbb{Z}_2[x]$—the ring of polynomials over $\mathbb{Z}_2$.

**Theorem 5.5.** *The ring $\mathbb{Z}_2[x]$ is a commutative, positive ring.*

*Proof.* Consider the polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$. The product of these polynomials is:

$$\begin{aligned} f(x) \cdot g(x) &= \sum_{i=0}^{m+n} \left( \sum_{i=j+k} a_j \cdot b_k \right) x^i \\ &= \sum_{i=0}^{m+n} \left( \sum_{i=j+k} b_k \cdot a_j \right) x^i \\ &= \sum_{i=0}^{m+n} \left( \sum_{i=j+k} b_j \cdot a_k \right) x^i \\ &= g(x) \cdot f(x) \end{aligned} \tag{5.31}$$

And, for any polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m \in \mathbb{Z}_2[x]$, we have that:

$$\begin{aligned} f(x) + f(x) &= (a_0 + a_0) + (a_1 + a_1)x + (a_2 + a_2)x^2 + \cdots + (a_m + a_m)x^m \\ &= 0 + 0x + 0x^2 + \cdots + 0x^m \\ &= 0 \end{aligned} \tag{5.32}$$

which concludes the proof. □

So, by theorem 5.2, the Clifford algebras over $\mathbf{R} = \mathbb{Z}_2[x]$ are commutative.

Shortly after discovering this example of a commutative, positive ring, I found a plethora of others. As it happens, any Clifford algebra over a commutative, positive ring is itself a commutative, positive ring.

**Theorem 5.6.** *The Clifford algebra over a commutative, positive ring is a commutative, positive ring.*

*Proof.* By theorem 5.2, we know that the Clifford algebra over a commutative, positive ring is commutative.

And, in the proof of theorem 5.2, each term contributes something which is its own additive inverse. It remains to show that if $\alpha+\alpha = 0$ and $\beta+\beta = 0$, that $(\alpha+\beta)+(\alpha+\beta) = 0$. This follows directly from the fact that addition is associative and commutative.

$$\begin{aligned} (\alpha + \beta) + (\alpha + \beta) &= \alpha + \beta + \alpha + \beta \\ &= \alpha + \alpha + \beta + \beta \\ &= (\alpha + \alpha) + (\beta + \beta) \\ &= 0 + 0 \\ &= 0 \end{aligned} \tag{5.33}$$

So, we have shown both commutativity and positive-ness. □

In particular, this means that if $\mathbf{R}$ is a commutative, positive ring, then the Clifford algebras commute over ${}_{\mathbf{R}}\mathbb{R}$, ${}_{\mathbf{R}}\mathbb{C}$, ${}_{\mathbf{R}}\mathbb{H}$, ${}_{\mathbf{R}}\mathcal{C}\ell_n$, ${}_{\mathbf{R}[x]}\mathcal{C}\ell_n$, ${}_{{}_{\mathbf{R}}\mathcal{C}\ell_n}\mathcal{C}\ell_m$, and the whole gamut of these sorts of combinations and their subalgebras.

5.3. **Over Other Rings.** From what we have seen in the proof of theorem 5.2 with regard to the terms that make up a product in the Clifford algebras, the commutative, positive rings are slightly more than is necessary to induce commutativity.

As we saw in the proof of theorem 5.2, the two terms $\mathbf{a} = \alpha\mathbf{e}_{a_1a_2\ldots a_s}$ and $\mathbf{b} = \beta\mathbf{e}_{b_1b_2\ldots b_t}$ will contribute $\pm(\alpha\beta)\mathbf{e}_{c_1c_2\ldots c_u}$ and $\pm(\beta\alpha)\mathbf{e}_{c_1c_2\ldots c_u}$ to the final products $\mathbf{ab}$ and $\mathbf{ba}$. As such, the real requirement to induce commutativity in $\mathcal{C}\ell_n$ is that the ring elements satisfy:

$$\pm(\alpha\beta) = \pm(\beta\alpha) \tag{5.34}$$

On the surface, equation 5.34 looks exactly like the requirement that it be commutive and positive. Definitely, the portion that $+(\alpha\beta) = +(\beta\alpha)$ requires that the ring be commutative. But, if the ring does not have a multiplicative identity, then there may exist elements which do not have to be their own additive inverses because they will never be the result of a product.

One example of such a ring is:

$$\begin{array}{c|cccc} + & 0 & a & b & c \\ \hline 0 & 0 & a & b & c \\ a & a & c & 0 & b \\ b & b & 0 & c & a \\ c & c & b & a & 0 \end{array} \qquad \begin{array}{c|cccc} \cdot & 0 & a & b & c \\ \hline 0 & 0 & 0 & 0 & 0 \\ a & 0 & c & c & 0 \\ b & 0 & c & c & 0 \\ c & 0 & 0 & 0 & 0 \end{array} \tag{5.35}$$

Here, neither $a$ nor $b$ are their own additive inverse. But, every product is either $c$ or 0 which are their own additive inverses.

However, if the ring had an identity, then either $\alpha$ or $\beta$ could be the identity. That would then require that every element of the ring be its own additive inverse.

## 6. Inducing In Other Algebras

The commutative, positive rings do not generally induce commutativity in the matrix algebra with basis $\{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$. What is different here?

In the case of the Clifford algebras, the terms $\mathbf{a} = \alpha\mathbf{e}_{a_1a_2\ldots a_s}$ and $\mathbf{b} = \beta\mathbf{e}_{b_1b_2\ldots b_t}$ contribute to (though not always with the same sign) the same $\mathbf{e}_{c_1c_2\ldots c_u}$ for both $\mathbf{ab}$ and $\mathbf{ba}$.

For this matrix algebra, that is not the case. If we let $\mathbf{a} = \alpha\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ and $\mathbf{b} = \beta\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, then we have

$$\mathbf{ab} = (\alpha\beta)\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] \tag{6.36}$$

and

$$\mathbf{ba} = (\beta\alpha)\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right] \tag{6.37}$$

So, the term **ab** contributes to the $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ term while the term **ba** contributes nothing.

The matrix algebra with basis $\{\mathbf{e}_1 = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \mathbf{e}_2 = \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\}$ fairs a bit better. In this case, the terms $\mathbf{a} = \alpha\mathbf{e}_1$ and $\mathbf{b} = \beta\mathbf{e}_2$ contribute something for both products **ab** and **ba**. Unfortunately, they do not both contribute to the coefficients of the same vector.

$$\mathbf{ab} = (\alpha\beta)\mathbf{e}_2 \tag{6.38}$$

but

$$\mathbf{ba} = (\beta\alpha)\mathbf{e}_1 \tag{6.39}$$

For both of these matrix algebras, the only finagling of $\alpha$ and $\beta$ which will ensure commutativity is the trivial case where the ring product is identical to zero.

This points to a necessary condition for inducing commutativity on a non-commutative algebra with some non-trivial ring. For any two basis elements $\mathbf{e}_i$ and $\mathbf{e}_j$ of the algebra, the products $\mathbf{e}_i\mathbf{e}_j$ and $\mathbf{e}_j\mathbf{e}_i$ must, at the very least, both contribute to the coefficients of the same basis vectors.

## 7. Summary of Results

The multiplicative structure of the basis elements of the Clifford algebras allows one to induce commutativity in the algebra by taking the algebra over certain rings. For the Clifford algebras, these rings are precisely those commutative rings $\mathbf{R}$ where for any $\alpha, \beta \in \mathbf{R}$ we have that $\alpha \cdot \beta + \alpha \cdot \beta = 2 \cdot \alpha \cdot \beta = 0$. In this paper, we demonstrated a plethora of commutative rings that meet this requirement.

Additionally, we explored some non-commutative algebras which could not be induced to commute over any non-trivial rings. We showed that, at the very least, any two basis elements $\mathbf{e}_i$ and $\mathbf{e}_j$ of the algebra must contribute to the same coefficients when multiplied $\mathbf{e}_i\mathbf{e}_j$ and $\mathbf{e}_j\mathbf{e}_i$. Future work on this topic could reveal other requirements on the algebra. For example, if $\mathbf{e}_i\mathbf{e}_j = \rho\mathbf{e}_j\mathbf{e}_i$, can commutativity still be induced if $\rho \notin \{-1, 0, 1\}$?

## 8. Concluding Remarks

I feel I rather thoroughly explored the requirements on the ring of coefficients for the Clifford algebras. But, I wish I had more time to explore the requirements on other algebras and to formulate more of the necessary and some of the sufficient conditions of the underlying algebras.

I read almost all of the [Lounesto] book and most of the [Dixon] book in preparing to write this paper. Both of them are very interesting. The [Dixon] book assumes a great deal of background knowledge in quantum mechanics and algebra that I just do not have. But, the [Lounesto] book has a great deal to offer that I think any advanced student could grasp during or after the 532 course. It also has some tacit assumptions about the reader's knowledge of quantum mechanics, but all of that content can be skimmed without losing the cool information on the Clifford algebras.

With respect to the content of this paper itself, I went through many ups and downs over the course of the quarter. One day I would be excited about how much there is to explore in this topic. The next day I would have a revelation that would sum up all that I had thought about the topic into a few lines. The next day, I would have a new revelation about another facet. In all, I think it came out to be just about the right size topic for one quarter.

## References


[Lounesto] Lounesto, Pertti. *Clifford Algebras and Spinors*, London Mathematical Society Lecture Note Series, No 239. Cambridge: Cambridge University Press, 1997. ISBN 0-521-59916-4

[Dixon] Dixon, Geoffrey M. *Division Algebras: Octonions, Quaternions, Complex Number and the Algebraic Design of Physics.* Dordrecht: Kluwer Academic Publishers. 1994. ISBN 0-7923-2890-6

[Weisstein] Weisstein, Eric. "Eric Weisstein's World of Mathematics". `http://mathworld.wolfram.com/` (regularly).

[Gull] Gull, Stephen, Anthony Lasenby, and Chris Doran. "Imaginary Numbers are not Real—The Geometric Algebra of Spacetime". 9 February 1993. `http://www.mrao.cam.ac.uk/~clifford/introduction/intro/intro.html` (24 April 2000).

[Parker] Parker, Sybil P., ed. *McGraw-Hill Dictionary of Mathematics.* New York: McGraw-Hill, 1994. ISBN 0-07-052433-5

[Hamermesh] Hamermesh, Morton. *Group Theory and Its Application to Physical Problems.* New York: Dover Publications, Inc. 1962. ISBN 0-486-66181-4

[Gilbert] Gilbert, Jimmie & Linda. *Elements of Modern Algebra.* Pacific Grove: Brooks/Cole. 2000. ISBN 0-534-37351-8



Center for Imaging Science, Rochester Institute of Technology, Rochester, NY 14623

*E-mail address*: `pat@csh.rit.edu`