WHY CAN'T I HAVE COUNTABLY MANY PAIRS OF SOCKS?

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Abstract. According to all sources, the Axiom of Choice is not required for a finite collection of nonempty sets yet *is* required for even a denumerable infinite collection. The author still does not comprehend why that is, but has assembled here all of the pieces.

Contents

1. The Problem

According to [Schecter], Bertrand Russell once said:

> To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed.

[Schecter] goes on to say that if there were only finitely many pairs of socks, we could simply use the definition of "nonempty" a finite number of times.

The author has endeavoured to gather, in this paper, all of the pieces required to see why there is a difference between a finite collection of pairs of socks and a countably infinite collection. The reader will hopefully fair better than the author has in assembling these pieces into an understanding.

Without much substantiation, [McGough] states that the Axiom of Choice is *not* required to choose one element from:

(1) a finite set
(2) an infinite set
(3) each member of an infinite set of singletons
(4) each member of an infinite set of shoes
(5) a finite set of sets if each of the members is infinite
(6) each member of an infinite set of sets of rationals
(7) each member of an infinite set of finite sets of reals

Results 1, 2, and 5 are all a consequence of the same theorem which is highlighted below in section 3. For result 3, the choice set is the union of the collection. Result

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4 is a consequence of the Axiom of Replacement since one and only one member of each pair will have the property "is a right shoe". Skipping result 6, for a moment, result 7 can also be achieved because each finite set of reals will have a least element. As such, one can use the Axiom of Replacement.

However, the Axiom of Choice *is* required to choose one element from:

(1) each member of an infinite set of socks
(2) an infinite set of sets if each of the members is infinite
(3) a denumerable set of sets if each of the members is infinite
(4) a denumerable set of sets if each of the members is denumerable
(5) each member of an infinite set of sets if each of the members is finite.
(6) each member of an infinite set of sets of reals
(7) each member of an infinite set of two-element sets whose members are sets of reals

About result 3, [McGough] says:

> In Cohen's model N there is a denumerable (in the model) set of sets of real numbers which has no choice function [1966, p. 139].

And, about result 7, [McGough] says:

> In Cohen's model N of ZF, AC fails for pairs of elements of $\mathcal{P}(\Re)$ [1966, p. 142].

But, this author has not been able to track down the bibliography for [McGough]. Most of the other results are left unexplained.

This author had suspected that result 6 from the first list was a consequence of the fact that the rationals were denumerable. Thus, given a set of rationals, it is possible to pick the first rational from the set using the Axiom of Replacement as one did to pick the minimum element from a finite set of reals. However, this author cannot see why such an approach would not work for any denumerable set, thus making result 4 from the second list achievable without the Axiom of Choice. This problem is revisted in section 5 below.

2. The Axiom of Choice

The Axiom of Choice can be stated in many ways. In [Devlin] p. 56, it is expressed:

Axiom 1. *Let $\mathcal{F}$ be a set of pairwise-disjoint, nonempty sets. Then, there is a set M that consists of precisely one element from each member of $\mathcal{F}$. The set M is called a choice set for $\mathcal{F}$.*

This is an inherently nonconstructive result. It claims only the existence of such a set. It does not specify a method by which one could attain the choice set. In Zermelo's own words, the Axiom of Choice, "as its wording, by the way, makes sufficiently clear, should be regarded as a pure *axiom of existence* (Existenzaxiom)." [Sierpinski] p. 96.

One can imagine sets from which one would be hard pressed to chose an element. For example, the Lucas-Lehmer test can show that a Mersenne number is composite without giving any hint as to what the factors of the number will be. Thus, one can imagine the set of factors of a composite Mersenne number and have no means by which to pick a member from that set.

However, this same problem occurs with other of the Zermelo–Fraenkel Axioms. For example, using the set of factors of a composite Mersenne number again, one can imagine applying the Power Set Axiom (as expressed in [Devlin] p. 40):

Axiom 2. *If x is a set, there is a set that consists of all and only the subsets of x.*

One is in an even bigger quandry than they had been in specifying a single member of the set.

3. Choice Functions for Finite Collections

There is a proof by induction in [Zuckerman] p. 386 that any finite collection of nonempty sets has a choice function even without the Axiom of Choice. First, one begins with the base case for the induction where the collection X contains only one set Y. The set Y must be nonempty.

$$X = \{Y\}$$

It is clear that the Cartesian product of the collection X and the elements in the sets of X is nonempty.

$$X \times \cup X = X \times Y$$

Each member $\langle Y, y\rangle$ of this Cartesian product is a choice function for the collection X. One needs only to pick one of the choice functions and evaluate it to attain a choice set for the collection X.

Now, one hopes to prove that given a choice function f for a collection X which contains n sets (each nonempty), one can obtain a choice function for a collection X' which contains $(n+1)$ sets (each nonempty). Let g be a choice function on the set $\left\{Y_{(n+1)}\right\}$. Then, it is easy to verify that if

$$X = \{Y_1, Y_2, Y_3, \ldots, Y_n\}$$

and

$$X' = \left\{Y_1, Y_2, Y_3, \ldots, Y_n, Y_{(n+1)}\right\} = X \cup \left\{Y_{(n+1)}\right\}$$

then $f \cup g$ is a choice function on the new collection X'.

From the above proof, it is clear that every finite collection of nonempty sets has a choice function. But, this author sees no reason why this induction cannot be extended to all ordinals. Yet, [Sierpinski] p. 97 states, without support, that this cannot be done "in a general way for every non-empty set" nor can it be done "for sets forming certain given sets of sets, e.g. for all non-empty sets of real numbers." The author shall return to this in section 5 below.

4. Choice Functions for Infinite Collections of n-Element Sets

Both [Sierpinski] p. 101 and [Zuckerman] p. 149 follow Mostowski and use the notation $[n]$ to refer to special cases of the axiom of choice. As stated in [Zuckerman]:

Axiom 3. *For each $n \in \mathbf{N}$, let $[n]$ be the following* ***Axiom of Choice for n-element sets****: Let X be a nonempty set of n-element sets. Then there is a function F mapping X into $\cup X$ with the property that $F(Y) \in Y$ for each Y in X.*

[Zuckerman] leaves it an exercise to show that the special case [1], the Axiom of Choice for 1-element sets, is a theorem in Zermelo–Fraenkel without the Axiom of Choice. As mentioned in section 1, $F(Y) = \cup Y$ suffices, giving the choice set $\cup X$.

[Sierpinski] proves that $[2] \rightarrow [4]$. The proof of this is rather interesting. This proof follows [Sierpinski]. Given a 4-element set, $A = \{a_1, a_2, a_3, a_4\}$, one can specify all six of the 2-element subsets of A. Call the collection of those subsets A^*. Because [2] is assumed, there is a function f that selects one member from each set of A^*. If we count the elements of A^* for which f selects a_i and call this n_i, then it is clear that

$$n_1 + n_2 + n_3 + n_4 = 6$$

From this, the n_i cannot all be equal. The elements of A can be relabeled so that

$$n_1 \leq n_2 \leq n_3 \leq n_4$$

Let B be the set of elements for which $n_i = n_1$. The set B can have either 1, 2, or 3 elements. If it has 1 element, define $g(A)$ to be that element. If it has 3 elements, define $g(A)$ to be the single element in $(A - B)$. And, if B has two elements, then define $g(A)$ to be $f(B)$. The relabelling of the elements of A to sort the n_i was necessary to eliminate any dependence of $g(A)$ on the particular labelling scheme chosen for A.

The choice function f of [2] was used to generate the choice function g of [4]. [Sierpinski] also proves that $[kn] \to [n]$. From this and the previous result, it follows that $[2] \equiv [4]$. Additionally, [Sierpinski] proves that if n is a natural number and $[p]$ is true for all prime $p \leq n$, then $[k]$ is true for every composite number $k \leq 2n + 1$.

Of course, as noted with socks above, the Zermelo–Fraenkel axioms without the Axiom of Choice are insufficient to prove [2]. And, in fact, [1] is the only case for which the Zermelo–Fraenkel axioms without the Axiom of Choice are sufficient.

5. The Author's Struggle with Countable Collections

As mentioned in section 3 above, this author does not understand why the induction proof that every finite collection of nonempty sets has a choice function does not extend to transfinite induction.

An insight is given in [Beachy] p. 212:

> The principle of induction can be extended to sets other than the natural numbers, but it requires the use of the axiom of choice, again in the form of Zorn's lemma.

[Beachy] then proves the principle of transfinite of induction. He starts with a subset A of a well-ordered set $(S, \leq)$. From there, he says that $A = S$ if

$$((\forall s < x)(s \in A)) \to (x \in A) \tag{1}$$

The proof is by contradiction. Assuming $A \neq S$, then there must be a least element of $S - A$. Call this element x. However, $x \in A$ by equation 1. This contradiction proves the principle.

This author, however, does not see why Zorn's lemma was needed here. This author thought the fact that S was well-ordered obviated the need for Zorn's lemma. Zorn's lemma is certainly applicable, but the author thought it only necessary for results on partial orders. In fact, [Devlin] p. 17 proves this same theorem (calling it "Induction on a Well-Ordering") in the same manner without reference to the Axiom of Choice or Zorn's Lemma.

The author admits that there are some non-intuitive results involving uncountable sets which bring into question the possibility of constructing a choice set for all infinite sets. For example, Aronszajn demonstrated a tree of height ω_1 with a countable number of nodes at each level which has no uncountable branch ([Devlin]

p. 113). The absence of an uncountable branch takes away one easy way to construct a choice set from the collection of levels in the tree. However, it still does not rule out a choice set. Indeed, since the nodes at each level of Aronszajn's tree are directly mapped to rationals, [McGough] assures us that there is a choice set even without the Axiom of Choice. (Recall that [McGough] claimed that one could choose one element from each member of an infinite set of sets of rationals.)

This author is reluctant to accept the nonconstructive view that one can choose a member from any nonempty set. But, [Sierpinski] p. 97 says that even those "who reject the axiom of choice in its general form do not question it in the case of a finite number of sets." Even accepting this, the author is still at a lost as to why the result does not extend to any well-ordered collection of sets.

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