PLANETARY MOTION IN FOUR DIMENSIONS

PATRICK STEIN

1. Introduction

The only independent vector quantities involved in a planetary orbit are the vector from the sun to the planet ($\vec{\mathbf{r}}$) and the velocity of the planet ($\vec{\mathbf{v}}$). All other vector quantities are derived from these and from various scalar quantities (the mass of the sun (M), the mass of the planet (m), etc.). These two vectors determine a plane. Without loss of generality, we can assume this happens in the x-y plane. As we shall see, there is nothing in the orbital mechanics which would cause the planet to move out of that plane. This is particularly important because much of the math involves cross-products which cannot really be done on four-dimensional vectors. As such, the math that follows requires that we be able to keep the orbit in a plane.

For ease of reference, the following derivations will follow those found on Erik Max Francis's web pages about orbital mechanics.[1]

2. Torque

Torque ($\vec{\tau}$) is the impressed force an a lever arm. It can be calculated as:

$$\vec{\tau}(t) = \vec{\mathbf{r}}(t) \times \vec{\mathbf{F}}(t) \tag{2.1}$$

The only force that the sun can impress upon the planet is that of gravitational attraction.

We can rewrite the vector from the sun to the planet ($\vec{\mathbf{r}}$) in terms of the magnitude of that vector (r) and its direction ($\hat{\mathbf{r}}$) as follows:

$$\vec{\mathbf{r}}(t) = r(t)\hat{\mathbf{r}}(t) \tag{2.2}$$

In four dimensions, the force of gravity must radiate in all four dimensions. As such, the surfaces of constant gravitational pull will be hyperspherical shells centered at the sun. The surface area of the hyperspherical shell is directly proportional to the cube of the radius. As such, the amount of force at any given point on a hyperspherical shell is inversely proportional to the cube of the radius. Thus, in four dimensions, the force exerted by a sun of mass M on a planet of mass m will be:

$$\vec{\mathbf{F}}(t) = -\frac{GmM}{r(t)^3}\hat{\mathbf{r}}(t) \tag{2.3}$$

Where G is the four-dimensional gravitational constant, r is the distance from the sun to the planet, and $\hat{\mathbf{r}}$ is a unit vector in the direction from the sun to the planet.

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[1] `http://www.alcyone.com/max/physics/kepler/1.html`

Using equations 2.2 and 2.3 in equation 2.1, we get:

$$\begin{aligned}\vec{\tau}(t) &= \vec{\mathbf{r}}(t) \times \vec{\mathbf{F}}(t) \\ &= (r(t)\vec{\mathbf{r}}(t)) \times \left(-\frac{GmM}{r(t)^3}\hat{\mathbf{r}}(t)\right) \\ &= \left(-\frac{GmM}{r(t)^2}\right)(\hat{\mathbf{r}}(t) \times \hat{\mathbf{r}}(t)) \\ &= \left(-\frac{GmM}{r(t)^2}\right)\vec{\mathbf{0}} \\ &= \vec{\mathbf{0}}\end{aligned} \tag{2.4}$$

As in the three-dimensional case, this makes perfect sense.

3. Conservation of angular momentum

Torque is the derivative of angular momentum.

$$\vec{\tau}(t) = \frac{d\vec{\mathbf{L}}}{dt} \tag{3.1}$$

From the previous section, we know that the torque is always zero. As such, the derivative of angular momentum must be zero.

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{0}} \tag{3.2}$$

Thus, the angular momentum must be a constant.

$$\vec{\mathbf{L}}(t) = \vec{\mathbf{L}}_0 \tag{3.3}$$

However, using equation 3.1 in equation 2.1, we can see that:

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}}(t) \times \vec{\mathbf{F}}(t) \tag{3.4}$$

From there, we can use Newton's second law of motion which says that the force ($\vec{\mathbf{F}}$) is equal to the mass (m) times the acceleration ($\vec{\mathbf{a}}$):

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}}(t) \times (m\vec{\mathbf{a}}(t)) \tag{3.5}$$

And, by definition, the acceleration of a body ($\vec{\mathbf{a}}$) is the time-derivative of its velocity ($\vec{\mathbf{v}}$).

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} \tag{3.6}$$

Using equation 3.6 in equation 3.5, we get:

$$\begin{aligned}\frac{d\vec{\mathbf{L}}}{dt} &= \vec{\mathbf{r}}(t) \times \left(m\frac{d\vec{\mathbf{v}}}{dt}\right) \\ &= m\left(\vec{\mathbf{r}}(t) \times \frac{d\vec{\mathbf{v}}}{dt}\right)\end{aligned} \tag{3.7}$$

Solving this differential equation, we see that:

$$\vec{\mathbf{L}}(t) = m\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{v}}(t)\right) \tag{3.8}$$

Technically, there would also be a constant of integration. In our case, however, we are not concerned with the actual value of the angular momentum. We are only concerned that it stay constant—that it is conserved. As such, we are just ignoring the constant term altogether.

Now, we're going to ignore the vector portion of the angular momentum and simply concentrate on its magnitude.

$$\begin{aligned} L(t) &= \left|\vec{\mathbf{L}}(t)\right| \\ &= \left|m\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{v}}(t)\right)\right| \\ &= m\left|\vec{\mathbf{r}}(t) \times \vec{\mathbf{v}}(t)\right| \end{aligned} \tag{3.9}$$

4. Kepler's Second Law

What we're looking for is the area swept out by the radius vector in a short time interval (Δt).

$$\Delta\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t) \tag{4.1}$$

Those three vectors form a triangle. During the short time (Δt), the area of the triangle is approximately equal to the area of the arc swept out.

The small area ($\Delta A(t)$) can be written as half of the area of the parallelogram defined by the radius vector ($\vec{\mathbf{r}}(t)$) and the change in the radius vector ($\Delta\vec{\mathbf{r}}(t)$).

$$\Delta A(t) = \frac{1}{2}\left|\vec{\mathbf{r}}(t) \times \Delta\vec{\mathbf{r}}(t)\right| \tag{4.2}$$

Dividing both sides by the small amount of time involved (Δt), we get:

$$\begin{aligned} \frac{\Delta A(t)}{\Delta t} &= \frac{1}{2}\left(\frac{1}{\Delta t}\right)\left|\vec{\mathbf{r}}(t) \times \Delta\vec{\mathbf{r}}(t)\right| \\ &= \frac{1}{2}\left|\frac{\vec{\mathbf{r}}(t) \times \Delta\vec{\mathbf{r}}(t)}{\Delta t}\right| \\ &= \frac{1}{2}\left|\vec{\mathbf{r}}(t) \times \frac{\Delta\vec{\mathbf{r}}(t)}{\Delta t}\right| \end{aligned} \tag{4.3}$$

If we take the limit as the small amount of time goes to zero in equation 4.3, we get:

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}\left|\vec{\mathbf{r}}(t) \times \frac{d\vec{\mathbf{r}}}{dt}\right| \\ &= \frac{1}{2}\left|\vec{\mathbf{r}}(t) \times \vec{\mathbf{v}}(t)\right| \end{aligned} \tag{4.4}$$

But, from equation 3.8, we know that:

$$\left|\vec{\mathbf{r}}(t) \times \vec{\mathbf{v}}(t)\right| = \frac{\vec{\mathbf{L}}(t)}{m} \tag{4.5}$$

And therefore:

$$\frac{dA}{dt} = \frac{\vec{\mathbf{L}}(t)}{2m} \tag{4.6}$$

That is, the time-rate-of-change of the area swept out by the planet is the magnitude of the angular momentum divided by twice the mass of the planet. And, we already know that the angular momentum is constant. Thus, the area swept out by the planet over any fixed time interval is constant regardless of where in the orbit this happens.

It should be noted that so far, this doesn't require gravity to have any particular magnitude. It requires only that gravity act radially from the center of the sun. Thus, the fact that gravity is an inverse-cube law in four dimensions whilst only an inverse-square law in three dimensions has not affected Kepler's second law. The planet must still sweep out equal areas in equal times.

5. Polar basis vectors

Following along with Erik Max Francis's development, we employ polar basis vectors. The first of these will be the basis vector for the radius.

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\hat{\mathbf{i}} + \sin\theta(t)\hat{\mathbf{j}} \tag{5.1}$$

As with the three-dimensional development, this is fortuitous since we're already making use of it in equation 2.2.

$$\vec{\mathbf{r}}(t) = r(t)\hat{\mathbf{r}}(t)$$

The second basis vector ($\hat{\theta}$) is always orthogonal to the radial basis vector ($\hat{\mathbf{r}}(t)$):

$$\hat{\theta}(t) = -\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\hat{\mathbf{j}} \tag{5.2}$$

Note that if we take the derivative of our second basis vector ($\hat{\theta}$) with respect to the angular position of the planet (θ), we see that:

$$\begin{aligned} \frac{d\hat{\theta}}{d\theta} &= -\cos\theta(t)\hat{\mathbf{i}} - \sin\theta(t)\hat{\mathbf{j}} \\ &= -\left(\cos\theta(t)\hat{\mathbf{i}} + \sin\theta(t)\hat{\mathbf{j}}\right) \\ &= -\hat{\mathbf{r}}(t) \end{aligned} \tag{5.3}$$

Then, working to expand velocity in terms of the basis vectors:

$$\begin{aligned} \vec{\mathbf{v}}(t) &= \frac{d\vec{\mathbf{r}}}{dt} \\ &= \frac{d(r\hat{\mathbf{r}})}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\frac{d\hat{\mathbf{r}}}{d\theta} \end{aligned} \tag{5.4}$$

Substituting in a variable for the angular velocity ($\omega(t) = \frac{d\theta}{dt}$), we have:

$$\vec{\mathbf{v}}(t) = \frac{dr}{dt}\hat{\mathbf{r}} + r(t)\omega(t)\frac{d\hat{\mathbf{r}}}{d\theta} \tag{5.5}$$

Finding a similar expression for the angular momentum ($\vec{\mathbf{L}}(t)$):

$$\begin{aligned}\vec{\mathbf{L}}(t) &= m\left|\vec{\mathbf{r}}(t)\times\vec{\mathbf{v}}(t)\right| \\ &= m\left|(r(t)\hat{\mathbf{r}}(t))\times\left(\frac{dr}{dt}\hat{\mathbf{r}}+r(t)\omega(t)\frac{d\hat{\mathbf{r}}}{d\theta}\right)\right| \\ &= m\left|(r(t)\hat{\mathbf{r}}(t))\times\left(\frac{dr}{dt}\hat{\mathbf{r}}\right)+(r(t)\hat{\mathbf{r}}(t))\times\left(r(t)\omega(t)\frac{d\hat{\mathbf{r}}}{d\theta}\right)\right| \\ &= mr(t)^2\omega(t)\left(\hat{\mathbf{r}}\times\hat{\theta}\right) \\ &= mr(t)^2\omega(t)\hat{\mathbf{k}}\end{aligned} \tag{5.6}$$

Then, looking at the magnitude of the angular momentum, we find that:

$$\begin{aligned}L(t) &= \left|\vec{\mathbf{L}}\right| \\ &= \left|mr(t)^2\omega(t)\hat{\mathbf{k}}\right| \\ &= mr(t)^2\omega(t)\left|\hat{\mathbf{k}}\right| \\ &= mr(t)^2\omega(t)\end{aligned} \tag{5.7}$$

6. Kepler's First Law

Now we get into using our inverse-cube law of gravity along with Newton's second law of motion.

$$\begin{aligned}m\vec{\mathbf{a}}(t) &= -\frac{GmM}{r(t)^3}\hat{\mathbf{r}} \\ \vec{\mathbf{a}}(t) &= -\frac{GM}{r(t)^3}\hat{\mathbf{r}}\end{aligned} \tag{6.1}$$

From equation 5.3, we know that:

$$\begin{aligned}\hat{\mathbf{r}}(t) &= -\frac{d\hat{\theta}}{d\theta} \\ &= -\frac{dt}{d\theta}\frac{d\hat{\theta}}{dt} \\ &= -\frac{1}{\omega(t)}\frac{d\hat{\theta}}{dt}\end{aligned} \tag{6.2}$$

Substituting those into equation 6.1, we get:

$$\begin{aligned}\vec{\mathbf{a}}(t) &= \frac{GM}{r(t)^3}\frac{1}{\omega(t)}\frac{d\hat{\theta}}{dt} \\ &= \frac{GM}{r(t)^3\omega(t)}\frac{d\hat{\theta}}{dt} \\ &= \frac{GmM}{mr(t)^3\omega(t)}\frac{d\hat{\theta}}{dt}\end{aligned} \tag{6.3}$$

And, substituting using equation 5.7, then we can write this as:

$$
\begin{aligned}
\vec{\mathbf{a}}(t) &= \frac{GmM}{L(t)r(t)}\frac{d\hat{\theta}}{dt} \\
\frac{L(t)r(t)}{GmM}\vec{\mathbf{a}}(t) &= \frac{d\hat{\theta}}{dt} \\
\frac{L(t)r(t)}{GmM}\frac{d\vec{\mathbf{v}}}{dt} &= \frac{d\hat{\theta}}{dt}
\end{aligned} \tag{6.4}
$$

Because angular momentum is conserved, $L(t)$ is a constant. Unfortunately, we must depart from the three-dimensional development. In the three-dimensional case, the $r(t)$ term is not present. It's also not constant. So, this integration isn't nice like it is in the three-dimensional case.

Now, as I have it, we can glom together all of the constants into one über-constant (α).

$$
\alpha = \frac{GmM}{L} \tag{6.5}
$$

With this and equation 5.3, we can rewrite equation 6.4

$$
\begin{aligned}
r(t)\frac{d\vec{\mathbf{v}}}{dt} &= \alpha\frac{d\hat{\theta}}{dt} \\
&= \alpha\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} \\
&= -\alpha\hat{\mathbf{r}}(t)\frac{d\theta}{dt} \\
&= -\alpha\left[\cos\theta(t)\hat{\mathbf{i}} + \sin\theta(t)\hat{\mathbf{j}}\right]\frac{d\theta}{dt}
\end{aligned} \tag{6.6}
$$

If only the $\hat{\mathbf{r}}(t)$ in the third step there were $r(t)\hat{\mathbf{r}}(t)$. Then, things would cancel a fair bit. Unfortunately, they don't cancel so very nicely.